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Diophantine approximation with primes and powers of two

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Abstract. We investigate the values taken by real linear combinations of two primes and a bounded number of powers of two. Under certain conditions, we are able to demonstrate that these values can be made arbitrarily close to any real number by taking sufficiently many powers of two.

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1. Introduction

It was conjectured by Goldbach in 1742 that every even integer exceeding two can be written as the sum of two prime numbers. Although this is widely believed to be true, a proof seems to be out of reach with the tools currently available. Nevertheless, some striking approximations to the conjecture have been established over the years. Vinogradov proved in 1937 that every sufficiently large odd integer is the sum of three primes, which of course would follow from the binary Goldbach conjecture. In the 1970s, Chen [4] proved that every sufficiently large even integer is the sum of a prime and an integer with at most two prime factors, and Montgomery and Vaughan [11] showed that the number of "exceptional" even integers up to X that are not the sum of two primes is $O(X^{1-\delta})$ for some $\delta > 0$. In a different direction, Linnik [10] showed that every sufficiently large even integer can be written as the sum of two primes and a bounded number of powers of two. This result is particularly striking, since the number of sums of powers of two up to X is only a

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power of $\log X$. In some sense, then, Linnik's result is a very strong approximation to Goldbach's conjecture. Using techniques developed by Gallagher [8], several workers have provided explicit bounds on the number of powers of two required. Very recently, Heath-Brown and Puchta [9] introduced a new approach leading to the conclusion that 13 powers of two suffice; moreover, they can reduce this number to 7 on assuming the Generalized Riemann Hypothesis. Independently, Pintz and Ruzsa [13] have obtained the same result on GRH and have further announced that they can prove the theorem unconditionally with only 9 powers of two.

Typically in additive number theory, when one has a method for handling a representation problem such as the one described above, it is of interest to examine to what extent the method can be applied to the analogous forms with real coefficients. For example, real analogues of the binary and ternary Goldbach problems have been examined in recent years by Vaughan [15], Brüdern, Cook, and Perelli [3], and Parsell [12]. Here we investigate the values taken by forms of the shape

$$\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{x_1} + \cdots + \mu_s 2^{x_s}. \tag{1}$$

We demonstrate, under certain conditions, that the values of such forms approximate any real number to arbitrary accuracy as s increases. Our main theorem is the following.

Theorem 1. *Suppose that λ_1 and λ_2 are real numbers such that λ_1/λ_2 is negative and irrational. Further suppose that $\mu_1, \dots, \mu_s$ are nonzero real numbers such that, for some i and j, the ratios λ_1/μ_i and λ_2/μ_j are rational. Finally, fix $\eta > 0$. Then there exists an integer s_0, depending at most on $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, and η, such that for every real number γ and every integer $s > s_0$, the inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{x_1} + \cdots + \mu_s 2^{x_s} + \gamma| < \eta \tag{2}$$

has infinitely many solutions in primes p_1 and p_2 and positive integers $x_1, \dots, x_s$.

Note that this falls short of a result to the effect that the values of the form (1) are dense in the real line for some particular value of s. Thus, as is often the case when attacking analogues of Waring's problem for forms with real coefficients (see for example [2] and [7]), this problem is in some ways more complicated than the coefficient-free version discussed above. On the other hand, certain aspects of our analysis are actually simpler because we have less need for information about the distribution of primes in arithmetic progressions.

We prove Theorem 1 using the Davenport-Heilbronn version of the Hardy-Littlewood method. Along the way, we make use of some of the new ideas of Heath-Brown and Puchta [9]. Because of the nature of the available estimates for exponential sums over powers of two, it is not possible to apply the method of Freeman [7] as in [12] to obtain an asymptotic lower bound for the number of solutions of the inequality (2). Rather, we will obtain a lower bound for the number of solutions only for a restricted sequence of values of the box size.

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2. The Davenport-Heilbronn method

We let X be a sufficiently large real number, and we write $L = \log_2(\varpi X/2M)$, where ϖ is a small positive number and $M = |\mu_1| + \cdots + |\mu_s|$. As usual, we write $e(y) = e^{2\pi i y}$ and introduce the exponential sums

$$f(\alpha) = \sum_{\varpi X \le p \le X} (\log p) e(\alpha p) \quad \text{and} \quad g(\alpha) = \sum_{1 \le x \le L} e(\alpha 2^x).$$

Here and throughout, the sum over p denotes summation over primes. We let $\mathcal{N}(X) = \mathcal{N}(X, \eta, \gamma, \boldsymbol{\lambda}, \boldsymbol{\mu})$ denote the number of integer solutions of the inequality (2) with p_1 and p_2 prime, $\varpi X \le p_1, p_2 \le X$, and $1 \le x_1, \dots, x_s \le L$.

In view of the hypotheses of Theorem 1, we may clearly suppose after rearranging variables that λ_1/μ_1 and λ_2/μ_2 are rational numbers. Furthermore, it clearly suffices to prove the theorem under the assumption that η is sufficiently small. Finally, after multiplying through by a suitable constant on both sides of (2) and possibly interchanging the roles of the first two indices, we may suppose that $\lambda_1 > 1$, that $\lambda_2 < -1$, and that $|\lambda_1/\lambda_2| \ge 1$.

We introduce the familiar kernel

$$K(\alpha) = \left(\frac{\sin \pi \alpha \eta}{\pi \alpha} \right)^2$$

and recall (see for example Baker [1], Lemma 14.1) that

$$\widehat{K}(t) = \int_{-\infty}^{\infty} e(\alpha t) K(\alpha)\, d\alpha = \max(0, \eta - |t|). \tag{3}$$

We also note that

$$K(\alpha) \ll \min(\eta^2, \alpha^{-2}). \tag{4}$$

When $\mathfrak{B} \subseteq \mathbb{R}$, we let

$$\mathcal{I}(X; \mathfrak{B}) = \int_{\mathfrak{B}} f(\lambda_1 \alpha) f(\lambda_2 \alpha) g(\mu_1 \alpha) \cdots g(\mu_s \alpha) e(\gamma \alpha) K(\alpha)\, d\alpha$$

and write $\mathcal{I}(X)$ for $\mathcal{I}(X; \mathbb{R})$. We will show that

$$\mathcal{I}(X) \gg \eta^2 X (\log X)^s \tag{5}$$

for some infinite sequence of values of X, where the implicit constant depends at most on $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. It follows from (3) that

$$\mathcal{I}(X) \le \eta (\log X)^2 \mathcal{N}(X),$$

so we can then deduce from (5) that

$$\mathcal{N}(X) \gg \eta X (\log X)^{s-2} \tag{6}$$

for this same sequence of values of X, which suffices to prove Theorem 1.

Let δ be a sufficiently small positive number, and write $P = X^{5/18 - \delta}$. We dissect the real line as follows. Write

$$\mathfrak{M} = \{\alpha : |\alpha| \le PX^{-1}\} \tag{7}$$

for the major arc,

$$\mathfrak{m} = \{\alpha : PX^{-1} < |\alpha| \le L^2\} \tag{8}$$

for the minor arcs, and

$$\mathfrak{t} = \{\alpha : |\alpha| > L^2\} \tag{9}$$

for the trivial arcs. Our plan of attack is to show that $\mathcal{I}(X;\mathfrak{t}) = o(XL^s)$, that $|\mathcal{I}(X;\mathfrak{m})| \leq C_1\eta XL^s$, and that $\mathcal{I}(X;\mathfrak{M}) \geq C_2\eta^2 XL^s$, where $\eta C_2 - C_1 \geq C_3\eta$ for some positive constant C_3.

3. Major arc asymptotics

We begin by recalling the familiar prime-counting functions

$$\vartheta(x) = \sum_{p\leq x} \log p \quad \text{and} \quad \psi(x) = \sum_{n\leq x} \Lambda(n),$$

where $\Lambda(n)$ denotes the von-Mangoldt function. We can express the exponential sum $f(\alpha)$ as the Riemann-Stieltjes integral

$$f(\alpha) = \int_{\varpi X}^{X} e(\alpha u) d\vartheta(u).$$

Furthermore, the explicit formula for $\psi(x)$ (see for example Davenport [6]), combined with the observation that $\psi(x) - \vartheta(x) \ll x^{1/2+\varepsilon}$ for each $\varepsilon > 0$, gives

$$\vartheta(x) = x - \sum_{|\gamma|\leq T} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - x^{-2}) + R(x,T),$$

where the sum is over zeros $\rho = \beta + i\gamma$ of the Riemann zeta function, where T is a parameter at our disposal, and where $R(x,T)$ is a piecewise differentiable function of x satisfying

$$R(x,T) \ll xT^{-1}\log^2(xT) + x^{1/2+\varepsilon}. \tag{10}$$

It follows that

$$f(\alpha) = v(\alpha) + w(\alpha) + E(\alpha), \tag{11}$$

where

$$v(\alpha) = \int_{\varpi X}^{X} e(\alpha u)\, du,$$

$$w(\alpha) = \sum_{|\gamma|\leq T} \int_{\varpi X}^{X} e(\alpha u) u^{\rho-1}\, du,$$

and

$$E(\alpha) = \int_{\varpi X}^{X} e(\alpha u) \frac{\partial}{\partial u}\left(R(u,T) - \frac{1}{2}\log(1-u^{-2})\right) du.$$

The estimate

$$v(\alpha) \ll \min(X, |\alpha|^{-1}) \tag{12}$$

is immediate, and a simple integration by parts using (7) and (10) gives

$$E(\alpha) \ll PXT^{-1}\log^2(XT) + PX^{1/2+\varepsilon}. \tag{13}$$

whenever $\alpha \in \mathfrak{M}$. In order to obtain estimates for $w(\alpha)$, we need to recall a zero-density estimate and a zero-free region for the Riemann zeta function.

Lemma 2. *Let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $\sigma \le \beta \le 1$ and $|\gamma| \le T$. For every $\varepsilon > 0$, one has*

$$N(\sigma, T) \ll T^{(12/5+\varepsilon)(1-\sigma)}.$$

Furthermore, there is a constant $c > 0$ such that the zeta function has no zeros with $|\gamma| \le T$ and $\beta > 1 - c(\log T)^{-2/3}$ whenever $T \ge 3$.

Proof. These results can be found in various sources. For example, the bound for $N(\sigma, T)$ follows from Lemma 3.4 of Ren [14], and the zero-free region follows from Lemma 4 of Vaughan [15]. □

Slightly sharper results are available in each case, but we will not need them. It follows from the argument leading to (3.11) of Ren [14] that

$$w(\alpha) \ll \min(X, X^{1/2}|\alpha|^{-1/2}) \sum_{|\gamma| \le T} X^{\beta-1}. \tag{14}$$

Using Lemma 2 and integration by parts, we find that

$$\sum_{|\gamma| \le T} X^{\beta-1} \ll \int_{1/2}^{1-\xi} X^{\sigma-1} dN(\sigma, T) \ll (\log X) \max_{1/2 \le \sigma \le 1-\xi} (X^{-1}T^{12/5+\varepsilon})^{1-\sigma},$$

where $\xi = c(\log T)^{-2/3}$. On choosing $T = X^{5/12-\varepsilon}$, we therefore obtain

$$\sum_{|\gamma| \le T} X^{\beta-1} \ll \exp(\log\log X - 2(\log X)^{1/4}) \ll \exp(-(\log X)^{1/4}).$$

Thus on recalling (11), (13), and (14), we have

$$f(\alpha) - v(\alpha) \ll \min(X, X^{1/2}|\alpha|^{-1/2}) \exp(-(\log X)^{1/4}) + X^{31/36},$$

provided that we choose ε to be sufficiently small. It now follows with a little calculation from (7) and (12) that for any real number μ we have

$$\begin{aligned}\int_{\mathfrak{M}} f(\lambda_1\alpha) f(\lambda_2\alpha) e(\alpha\mu) K(\alpha)\, d\alpha - J(X, \mu) &\ll X \exp(-(\log X)^{1/4}) \\ &\ll X(\log X)^{-s-1},\end{aligned} \tag{15}$$

where

$$J(X, \mu) = \int_{\mathfrak{M}} v(\lambda_1\alpha) v(\lambda_2\alpha) e(\alpha\mu) K(\alpha)\, d\alpha.$$

Furthermore, (7) and (12) imply that

$$J(X, \mu) = J(\mu) + O(X^{13/18+\delta}), \tag{16}$$

where

$$J(\mu) = \int_{-\infty}^{\infty} v(\lambda_1\alpha) v(\lambda_2\alpha) e(\alpha\mu) K(\alpha)\, d\alpha.$$

On interchanging the order of integration, we find that

$$J(\mu) = \int_{\varpi X}^{X} \int_{\varpi X}^{X} \widehat{K}(\lambda_1 u_1 + \lambda_2 u_2 + \mu)\, du_1\, du_2.$$

Suppose that $|\mu| \le \varpi X$ and that $2\varpi\lambda_1 X \le |\lambda_2| u_2 \le (1-\varpi)X$. Then (3) shows that there is an interval for u_1, of length η/λ_1 and contained in $[\varpi X, X]$, on which $\widehat{K}(\lambda_1 u_1 + \lambda_2 u_2 + \mu) \ge \eta/2$. It follows that

$$J(\mu) \ge \frac{(1-3\varpi\lambda_1)\eta^2}{2|\lambda_1\lambda_2|} X. \tag{17}$$

From (15) and (16), we obtain

$$\mathcal{I}(X;\mathfrak{M}) = \sum_{\boldsymbol{x}\in[1,L]^s} J(\mu_1 2^{x_1} + \cdots + \mu_s 2^{x_s} + \gamma) + O(X(\log X)^{-1}).$$

Moreover, our definition of L ensures that $|\mu_1 2^{x_1} + \cdots + \mu_s 2^{x_s} + \gamma| \le \varpi X$ whenever X is sufficiently large. Thus we deduce from (17) that

$$\mathcal{I}(X;\mathfrak{M}) \ge \frac{(1-\varpi')\eta^2}{2|\lambda_1\lambda_2|} XL^s \tag{18}$$

when X is sufficiently large. Here we have written $\varpi' = 4\lambda_1\varpi$, which can be taken to be arbitrarily small.

4. An auxiliary mean value estimate

A key ingredient in the treatment of the minor arcs is the following mean value estimate, which we derive following the method of Heath-Brown and Puchta [9].

Lemma 3. *Suppose that $\lambda/\mu = a/q$, where a and q are nonzero integers with $q > 0$ and $(a,q) = 1$. If $\eta < |\lambda/a|$ and X is sufficiently large, then one has*

$$\int_{-\infty}^{\infty} |f(\lambda\alpha)g(\mu\alpha)|^2 K(\alpha)\, d\alpha \le 25(\log 2q)\eta X L^2.$$

Proof. In view of (3), we have

$$\int_{-\infty}^{\infty} |f(\lambda\alpha)g(\mu\alpha)|^2 K(\alpha)\, d\alpha \le \eta(\log X)^2 S(X), \tag{19}$$

where $S(X)$ is the number of solutions of the inequality

$$|\lambda(p_1 - p_2) + \mu(2^{x_1} - 2^{x_2})| < \eta \tag{20}$$

with p_1 and p_2 primes not exceeding X and x_1 and x_2 integers not exceeding L. Since we have assumed that $\eta < |\lambda/a|$, the inequality (20) is equivalent to the equation

$$p_1 - p_2 + \frac{q}{a}(2^{x_1} - 2^{x_2}) = 0,$$

and here we may bound the number of solutions by following the argument of Heath-Brown and Puchta [9], Lemma 8. Let $r(n)$ denote the number of representations of n as a difference of two primes lying in the interval $[\varpi X, X]$. By a theorem of Chen [5], whenever $n \ne 0$ we have

$$r(n) \le 5.2\, h(n) X(\log X)^{-2},$$

where

$$h(n) = \prod_{p|n,\ p>2} \left(1 + \frac{1}{p-2}\right).$$

Thus, on separating out the diagonal solutions, we find that

$$\begin{aligned} S(X) &\le X + 2 \sum_{1\le x_1<x_2\le L} r\left(\frac{q}{a}(2^{x_2}-2^{x_1})\right) \\ &\le X + 10.4\, X(\log X)^{-2} \sum_{1\le x_1<x_2\le L} h\left(\frac{q}{a}(2^{x_2}-2^{x_1})\right). \end{aligned}$$

Obviously, only terms for which a divides $2^{x_2}-2^{x_1}$ contribute to the above sums. Since $h(n)$ is a multiplicative function, we have

$$h\left(\frac{q}{a}(2^{x_2}-2^{x_1})\right) \le h(q)h\left(\frac{2^{x_2}-2^{x_1}}{a}\right) \le h(q)h(2^{x_2-x_1}-1),$$

and a simple calculation shows that $h(q) \le 2\log 2q$. Thus we obtain

$$S(X) \le X + 20.8\, X \frac{\log 2q}{(\log X)^2} \sum_{1\le x\le L} (L-x)h(2^x-1). \tag{21}$$

By equations (34) and (41) of Heath-Brown and Puchta [9], we have

$$\sum_{1\le x\le Y} h(2^x-1) \le 2.2142\, Y + O(Y^{1/2}),$$

and partial summation therefore yields

$$\sum_{1\le x\le L} (L-x)h(2^x-1) \le 1.1072\, L^2 + O(L^{3/2}).$$

Combining this with (21) gives

$$S(X) \le (1+23.03\log 2q)X \le 24.5(\log 2q)X,$$

and the lemma now follows from (19) on noting that $\log X \le (1+\varpi)L$ for X sufficiently large. $\square$

5. The minor and trivial arcs

We need the following Weyl-type estimate for the exponential sum over primes, which we obtain via a standard argument (see for example [15], Lemma 11).

Lemma 4. *Suppose that λ_1/λ_2 is irrational, and let $X = q^2$, where q is the denominator of a convergent to the continued fraction for λ_1/λ_2. Then one has*

$$\sup_{\alpha\in\mathfrak{m}} |f(\lambda_1\alpha)f(\lambda_2\alpha)| \ll X^{15/8}(\log X)^5.$$

Proof. Let $\alpha \in \mathfrak{m}$, and let $Q = X^{1/4}L^{-2} \le P$. By Dirichlet's Theorem, there exist integers a_i and q_i with $1 \le q_i \le XQ^{-1}$ and $(q_i, a_i) = 1$ such that

$$|\lambda_i\alpha q_i - a_i| \le QX^{-1} \qquad (i = 1, 2).$$

Clearly, $a_1a_2 \ne 0$, since otherwise we would have $\alpha \in \mathfrak{M}$. Now suppose that $q_1 \le Q$ and $q_2 \le Q$. We have

$$a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2 = \frac{a_2}{\lambda_2\alpha}(\lambda_1\alpha q_1 - a_1) - \frac{a_1}{\lambda_2\alpha}(\lambda_2\alpha q_2 - a_2),$$

and it follows that

$$|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2| \le 2\left(1 + \left|\frac{\lambda_1}{\lambda_2}\right|\right) Q^2X^{-1} < \frac{1}{2}q^{-1}$$

for X sufficiently large. We therefore deduce from (8) and Legendre's law of best approximation that

$$X^{1/2} = q \le |a_2 q_1| \ll q_1 q_2 L^2 \le Q^2 L^2 \le X^{1/2} L^{-2}.$$

From this contradiction, we conclude that either $q_1 > Q$ or $q_2 > Q$, and Theorem 3.1 of Vaughan [16] therefore yields the estimate

$$f(\lambda_i \alpha) \ll L^4 (X q_i^{-1/2} + X^{4/5} + X^{1/2} q_i^{1/2}) \ll X^{7/8} (\log X)^5$$

for $i = 1$ or $i = 2$, and the lemma now follows on making a trivial estimate. □

It is not currently known how to obtain estimates of the above type for the exponential sums over powers of two. However, the following lemma provides non-trivial estimates except on a set of very small measure.

Lemma 5. *Let $\mathcal{A}_\nu$ denote the set of $\alpha \in \mathfrak{m}$ for which $|g(\mu_i \alpha)| \ge \nu L$ for some i with $1 \le i \le s$, and write*

$$F(\xi, h) = \frac{1}{2^h} \sum_{r=0}^{2^h - 1} \exp\left[\xi \operatorname{Re} \left(\sum_{x=0}^{h-1} e(r 2^{x-h}) \right) \right].$$

Then, for any $h \in \mathbb{N}$, $\xi > 0$, and $\varpi > 0$, one has $\operatorname{meas}(\mathcal{A}_\nu) \ll L^2 X^{-E(\nu)}$, *where*

$$E(\nu) = \frac{\xi \nu}{\log 2} - \frac{\log F(\xi, h)}{h \log 2} - \frac{\varpi \xi \nu}{\log 2}.$$

Proof. This follows from the proof of Lemma 1 of Heath-Brown and Puchta [9] on recalling (8) and considering the union of the sets $\{\alpha \in \mathfrak{m} : |g(\mu_i \alpha)| \ge \nu L\}$. □

Applying Lemma 5 with ϖ sufficiently small, $\xi = 1.55$, and $h = 10$, we find using Mathematica that $E(0.954) = 0.87553$ is admissible. From this point on, we set $\nu = 0.954$.

The minor arcs are dealt with in two pieces. First of all, by applying Lemma 4 and making trivial estimates, we obtain

$$\mathcal{I}(X; \mathcal{A}_\nu) \ll X^{15/8 - E(\nu)} L^{s+7} \ll X^{0.9995}.$$

Next, by the Cauchy-Schwarz inequality, we have

$$|\mathcal{I}(X; \mathfrak{m} \backslash \mathcal{A}_\nu)| \le (\nu L)^{s-2} \mathcal{J}_1^{1/2} \mathcal{J}_2^{1/2},$$

where

$$\mathcal{J}_i = \int_{-\infty}^{\infty} |f(\lambda_i \alpha) g(\mu_i \alpha)|^2 K(\alpha)\, d\alpha.$$

We therefore deduce from Lemma 3 that

$$|\mathcal{I}(X; \mathfrak{m} \backslash \mathcal{A}_\nu)| \le (0.954)^{s-2} C \eta X L^s, \tag{22}$$

where C is a constant depending on the denominators of the rational numbers λ_1/μ_1 and λ_2/μ_2. Specifically, if we denote these denominators by q_1 and q_2, then the lemma shows that we may take

$$C = 25 (\log 2q_1)^{1/2} (\log 2q_2)^{1/2}.$$

Finally, to handle the trivial arcs, we observe that (9) gives

$$\mathcal{I}(X;\mathfrak{t}) \ll L^s \int_{L^2}^{\infty} |f(\lambda_i \alpha)|^2 K(\alpha)\, d\alpha$$

for $i = 1$ or $i = 2$. On using (4) and making a change of variable, we find that

$$\mathcal{I}(X;\mathfrak{t}) \ll L^s \sum_{n \geq L^2} n^{-2} \int_n^{n+1} |f(\lambda_i \alpha)|^2 d\alpha \ll L^{s-2} \int_0^1 |f(\alpha)|^2 d\alpha \ll XL^{s-1}.$$

On comparing (18) and (22), and recalling our analysis of the sets $\mathcal{A}_\nu$ and $\mathfrak{t}$, we find that $\mathcal{I}(X) \gg \eta^2 XL^s$ for the sequence of X described above, provided that

$$s > 2 + \frac{\log(1 - 2\varpi')\eta - \log 2C|\lambda_1 \lambda_2|}{\log 0.954}.$$

This establishes (5), and hence (6), which completes the proof of Theorem 1.

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