ON SIMULTANEOUS DIAGONAL INEQUALITIES

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1. Introduction

Let $F_1, \dots, F_t$ be diagonal forms of degree k with real coefficients in s variables, and let τ be a positive real number. The solubility of the system of inequalities

$$|F_1(\mathbf{x})| < \tau, \ \dots \ , |F_t(\mathbf{x})| < \tau$$

in integers $x_1, \dots, x_s$ has been considered by a number of authors over the last quarter-century, starting with the work of Cook [9] and Pitman [13] on the case $t = 2$. More recently, Brüdern and Cook [8] have shown that the above system is soluble provided that s is sufficiently large in terms of k and t and that the forms $F_1, \dots, F_t$ satisfy certain additional conditions. What has not yet been considered is the possibility of allowing the forms $F_1, \dots, F_t$ to have different degrees. However, with the recent work of Wooley [18], [20] on the corresponding problem for equations, the study of such systems has become a feasible prospect. In this paper we take a first step in that direction by studying the analogue of the system considered in [18] and [20]. Let $\lambda_1, \dots, \lambda_s$ and $\mu_1, \dots, \mu_s$ be real numbers such that for each i either λ_i or μ_i is nonzero. We define the forms

$$\begin{aligned} F(\mathbf{x}) &= \lambda_1 x_1^3 + \cdots + \lambda_s x_s^3 \\ G(\mathbf{x}) &= \mu_1 x_1^2 + \cdots + \mu_s x_s^2 \end{aligned}$$

and consider the solubility of the system of inequalities

$$\begin{aligned} |F(\mathbf{x})| &< (\max |x_i|)^{-\sigma_1} \\ |G(\mathbf{x})| &< (\max |x_i|)^{-\sigma_2} \end{aligned} \tag{1.1}$$

in rational integers $x_1, \dots, x_s$. Although the methods developed in Wooley [19] hold some promise for studying more general systems, we do not pursue this in the present paper. We devote most of our effort to proving

Theorem 1. *Let $s \geq 13$, and let $\lambda_1, \dots, \lambda_s$ and $\mu_1, \dots, \mu_s$ be real numbers such that λ_i/λ_j and μ_i/μ_j are algebraic and irrational for some i and j. Then the simultaneous inequalities* (1.1) *have infinitely many solutions in rational integers provided that*

(a) *$F(\mathbf{x})$ has at least $s-4$ variables explicit,*
(b) *$G(\mathbf{x})$ has at least $s-5$ variables explicit,*
(c) *the simultaneous equations $F(\mathbf{x}) = G(\mathbf{x}) = 0$ have a non-singular real solution, and*
(d) *one has $\sigma_1 + \sigma_2 < \frac{1}{12}$.*

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If $\Theta(P)$ denotes the number of solutions of (1.1) with $\mathbf{x} \in [1, P]^s$, then our arguments will in fact show that $\Theta(P) \gg P^{s-5-\sigma_1-\sigma_2}$ as $P \to \infty$. We also note for future reference that condition (c) implies that the quadratic form G is indefinite, which is plainly a necessary requirement for solubility.

When either F or G has a large number of zero coefficients, we can exploit results for a single inequality to obtain

Theorem 2. *Let $\lambda_1, \ldots, \lambda_s$ and $\mu_1, \ldots, \mu_s$ be real numbers. The simultaneous inequalities* (1.1) *have infinitely many solutions in rational integers provided that*

(a) *$F(\mathbf{x})$ has at least 7 variables explicit,*
(b) *$G(\mathbf{x})$ has at least 5 variables explicit,*
(c) *the simultaneous equations $F(\mathbf{x}) = G(\mathbf{x}) = 0$ have a non-singular real solution, and*
(d) *one of the following holds:*
  (i) *at least 4 of the λ_i are zero and $\max(\sigma_1, \sigma_2) \le 10^{-5}$, or*
  (ii) *at least 7 of the μ_i are zero and $\sigma_1 \le 10^{-4}$.*

We remark that condition (b) is not actually needed to prove the stated version of Theorem 2; however, the condition arises naturally in discussing possible improvements on condition (d)(ii), so we state it for convenience.

In Section 2, we deduce Theorem 2 in an elementary manner from results on a single Diophantine inequality. We also consider a refinement of condition (d)(ii) which would follow from improvements in our understanding of cubic inequalities.

We then prove Theorem 1 in Sections 3, 4, and 5, using a two-dimensional version of the Davenport-Heilbronn method. We show that when P is sufficiently large one has

$$\Theta_s(P) \gg \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{H}(\boldsymbol{\alpha}) \mathcal{K}(\boldsymbol{\alpha}) \, d\boldsymbol{\alpha},$$

where $\mathcal{H}(\boldsymbol{\alpha})$ is a suitable product of exponential sums (many of which we restrict to smooth numbers) and $\mathcal{K}(\boldsymbol{\alpha})$ is a product of kernel functions. We then dissect the plane in analogy with the one-dimensional Davenport-Heilbronn method. The success of our minor arc analysis depends heavily on an estimate of Wooley [20] for the 10th moment of a certain exponential sum over smooth numbers and also on a result of R. Baker [2] relating the size of a certain exponential sum to the existence of good rational approximations to the coefficients of its argument. The treatment of the major arc is essentially straightforward using the ideas of Wooley [18].

Finally, in Section 6, we discuss the possibility of weakening some of the hypotheses imposed in Theorems 1 and 2.

Throughout our analysis, implicit constants in the notations of Vinogradov and Landau may depend on the coefficients $\lambda_1, \ldots, \lambda_s$ and $\mu_1, \ldots, \mu_s$, the exponents σ_1 and σ_2, and also on any parameters denoted by ε or δ.

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2. Forms with Many Zero Coefficients

Here we prove Theorem 2 using results on a single inequality. We first consider the case (d)(i). The argument is similar to that given in Lemmata 6.3, 6.4, and 6.5 of Wooley [18], but it also incorporates the recent work of Baker, Brüdern, and Wooley [3] on cubic inequalities in 7 variables and makes use of a result of Birch and Davenport [4] on small solutions of quadratic inequalities in 5 variables. We start with an analogue of [18], Lemma 6.3.

Lemma 2.1. *Suppose there is a rearrangement of the variables $x_1, \dots, x_s$ such that $\lambda_i = 0$ for $i = 1, \dots, 4$ and $\mu_1, \dots, \mu_4$ are not all of the same sign. Then Theorem 2 holds in the case (d)(i).*

Proof. Let $\sigma = 1.43 \times 10^{-4}$ and $\delta = \frac{1}{10}\sigma$. It is easily seen that the main theorem of [3] holds with the above value of σ, although the result is stated with a slightly smaller exponent. Thus by condition (a) of Theorem 2, there exist infinitely many $(s-4)$-tuples of integers $(a_5, \dots, a_s)$ such that

$$\left|\lambda_5 a_5^3 + \cdots + \lambda_s a_s^3\right| < (\max |a_i|)^{-\sigma}. \tag{2.1}$$

Now put $M_i = \mu_i$ for $i = 1, \dots, 4$, and put

$$M_5 = \mu_5 a_5^2 + \cdots + \mu_s a_s^2.$$

If $|M_5| < (\max |a_i|)^{-\delta}$, then we can take $x_1 = \cdots = x_4 = 0$ and $x_i = a_i$ for $i = 5, \dots, s$. Otherwise, by the main theorem of [4] we can find (for $\max |a_i|$ sufficiently large) integers $u_1, \dots, u_5$, not all zero, such that

$$\left|M_1 u_1^2 + \cdots + M_5 u_5^2\right| < (\max |a_i|)^{-\delta} \tag{2.2}$$

and

$$|M_1 u_1^2| + \cdots + |M_5 u_5^2| \ll (\max |a_i|)^{\delta(4+5\delta)} |M_1 \cdots M_5|^{1+\delta}.$$

But $M_5 \ll (\max |a_i|)^2$, so that

$$|u_j| \ll (\max |a_i|)^{1+\frac{\delta}{2}(6+5\delta)} \quad (j = 1, \dots, 4)$$

and

$$|u_5| \ll (\max |a_i|)^{\frac{\delta}{2}(6+5\delta)}.$$

Hence on putting $\mathbf{x} = (u_1, \dots, u_4, u_5 a_5, \dots, u_5 a_s)$, we have

$$\max |x_i| \ll (\max |a_i|)^{1+\frac{\delta}{2}(6+5\delta)}$$

and

$$|F(\mathbf{x})| < |u_5|^3 (\max |a_i|)^{-\sigma} \ll (\max |a_i|)^{\frac{3\delta}{2}(6+5\delta)-\sigma}.$$

Thus on taking

$$\varepsilon < \frac{2\sigma - 3\delta(6+5\delta)}{2 + \delta(6+5\delta)}$$

we see that for $\max |a_i|$ sufficiently large one has

$$|F(\mathbf{x})| < (\max |x_i|)^{-\varepsilon},$$

and so we may take $\sigma_1 = 1.429 \times 10^{-5}$. Moreover, on taking

$$\gamma < \frac{2\delta}{2 + \delta(6+5\delta)}$$

we have

$$|G(\mathbf{x})| < (\max |a_i|)^{-\delta} < (\max |x_i|)^{-\gamma}$$

for $\max|a_i|$ sufficiently large, so we may take $\sigma_2 = 1.429 \times 10^{-5}$. □

When the hypothesis of Lemma 2.1 is not satisfied, we need some additional control over the solution to our cubic inequality (2.1) in order to guarantee that the quadratic in (2.2) is indefinite. Specifically, we require the following analogue of [18], Lemma 6.4.

Lemma 2.2. *Let $\lambda_1, \ldots, \lambda_t$ $(t \geq 7)$ be non-zero real numbers, and suppose that $(\eta_1, \ldots, \eta_t)$ is a real solution of the equation*

$$\lambda_1 x_1^3 + \cdots + \lambda_t x_t^3 = 0$$

with $0 < \eta_i < 1$ for all i. Then for any $\alpha \in (0,1)$ and $P > P_0(\boldsymbol{\eta}, \boldsymbol{\lambda}, \alpha)$, there exist integers $y_1, \ldots, y_t$ such that

$$|\lambda_1 y_1^3 + \cdots + \lambda_t y_t^3| < (\max|y_i|)^{-\sigma},$$

where $\sigma = 1.43 \times 10^{-4}$ and

$$(1-\alpha)\eta_i P < y_i \leq (1+\alpha)\eta_i P \quad (i = 1, \ldots, t). \tag{2.3}$$

Proof. If the λ_i are all in rational ratio, then the result follows from Lemma 6.4 of [18]. Otherwise, we follow through the analysis of [3], restricting the ranges of summation on the generating functions so that only values of the variables satisfying (2.3) are included. All of the required estimates continue to hold, with only the major arc analysis requiring a slight modification. □

Now we can complete the proof of case (d)(i) by arguing as in [18], Lemma 6.5. Suppose that at least 4 of the λ_i are zero, and rearrange variables so that $\lambda_1, \ldots, \lambda_t \neq 0$ and $\lambda_i = 0$ for $i = t+1, \ldots, s$. By condition (c) and the argument of [18, Lemma 6.2], we may assume that the equations $F(\mathbf{x}) = G(\mathbf{x}) = 0$ have a real solution $(\eta_1, \ldots, \eta_s)$ with all of the η_i non-zero, and then on replacing λ_i by $-\lambda_i$ if necessary and using homogeneity we may assume that $0 < \eta_i < \frac{1}{2}$ for all i. Further, by Lemma 2.1, we may assume that $\mu_{t+1}, \ldots, \mu_s$ are all positive, so that

$$\mu_1\eta_1^2 + \cdots + \mu_t\eta_t^2 = -(\mu_{t+1}\eta_{t+1}^2 + \cdots + \mu_s\eta_s^2) = -C < 0.$$

Let α, P, and $(y_1, \ldots, y_t)$ be as in Lemma 2.2 with

$$\alpha < \frac{2C}{3t}\left(\max|\mu_i|\right)^{-1},$$

and put $M = \mu_1 y_1^2 + \cdots + \mu_t y_t^2$. Then

$$|M + CP^2| \leq P^2(\alpha^2 + 2\alpha) \sum_{1 \leq i \leq t} |\mu_i \eta_i^2| < \frac{1}{2}CP^2,$$

so that

$$M < -\frac{1}{2}CP^2 < 0.$$

Now let $\delta = 1.43 \times 10^{-5}$ as before. If $|M| < P^{-\delta}$, then we can take $x_i = y_i$ for $i = 1, \ldots, t$ and $x_{t+1} = \cdots = x_s = 0$. Otherwise, for P sufficiently large, we may use the result of [4] as in the proof of Lemma 2.1 to find integers $v_t, \ldots, v_s$, not all zero, with

$$|v_t| \ll P^{\frac{\delta}{2}(6+5\delta)} \quad \text{and} \quad |v_i| \ll P^{1+\frac{\delta}{2}(6+5\delta)} \quad (i = t+1, \ldots, s)$$

such that

$$|Mv_t^2 + \mu_{t+1}v_{t+1}^2 + \cdots + \mu_s v_s^2| < P^{-\delta}.$$

Proceeding exactly as in the proof of Lemma 2.1, we find that

$$\mathbf{x} = (y_1 v_t, \ldots, y_t v_t, v_{t+1}, \ldots, v_s)$$

satisfies (1.1) with $\sigma_1 = \sigma_2 = 10^{-5}$, and this completes the proof of Theorem 2 in the case (d)(i).

The case (d)(ii) of Theorem 2 follows immediately from the results of [3], and this completes the proof of the theorem.

We now investigate the possibility of reducing the number of zero coefficients required by condition (d)(ii) from 7 to 6, in accordance with [18] and [20]. Brüdern [7], improving on a result of Pitman and Ridout [14], has shown that if $\lambda_1, \ldots, \lambda_9$ are real numbers with $|\lambda_i| \geq 1$ for all i then there exist integers $x_1, \ldots, x_9$ satisfying

$$|\lambda_1 x_1^3 + \cdots + \lambda_9 x_9^3| < 1$$

and

$$0 < \sum_{i=1}^{9} |\lambda_i x_i^3| \ll_\delta |\lambda_1 \cdots \lambda_9|^{1+\delta}. \tag{2.4}$$

Unfortunately, in order to use this result in an argument like the one in Lemma 2.1 we would have to assume that $G(\mathbf{x})$ had at least eight zero coefficients, and in this situation we would do better to apply the results of [6]. Suppose, however, that the above result held with 7 variables instead of 9. Then condition (d)(ii) of Theorem 2 could be replaced by

(d)(ii)′ *at least* 6 *of the* μ_i *are zero and* $\max(\sigma_1, \sigma_2) \leq 10^{-2}$.

The argument resembles the one above, but an argument like the one ensuing from Lemma 2.2 will not be necessary since the quadratic under consideration there will be replaced by a cubic.

Proceeding just as in Lemma 2.1, we fix $\sigma < 1/10$ and $\delta = 1/70$. After rearranging variables, we may assume that $\mu_1 = \cdots = \mu_6 = 0$. Now by condition (b) of Theorem 2 and an easily obtained quantitative version of the classical Davenport-Heilbronn Theorem, we see that there exist infinitely many $(s-6)$-tuples of integers $(a_7, \ldots, a_s)$ such that

$$\left|\mu_7 a_7^2 + \cdots + \mu_s a_s^2\right| < (\max |a_i|)^{-\sigma}.$$

Now put $\Lambda_i = \lambda_i$ for $i = 1, \ldots, 6$, and put

$$\Lambda_7 = \lambda_7 a_7^3 + \cdots + \lambda_s a_s^3.$$

If $|\Lambda_7| < (\max |a_i|)^{-\delta}$, then we can take $x_1 = \cdots = x_6 = 0$ and $x_i = a_i$ for $i = 7, \ldots, s$. Otherwise, by our hypothesis, we can find (for $\max |a_i|$ sufficiently large) integers $u_1, \ldots, u_7$, not all zero, such that

$$\left|\Lambda_1 u_1^3 + \cdots + \Lambda_7 u_7^3\right| < (\max |a_i|)^{-\delta}$$

and

$$|\Lambda_1 u_1^3| + \cdots + |\Lambda_7 u_7^3| \ll (\max |a_i|)^{\delta(6+7\delta)} |\Lambda_1 \cdots \Lambda_7|^{1+\delta}.$$

But $\Lambda_7 \ll (\max|a_i|)^3$, so that

$$|u_j| \ll (\max|a_i|)^{1+\frac{\delta}{3}(9+7\delta)} \quad (j = 1, \dots, 6)$$

and

$$|u_7| \ll (\max|a_i|)^{\frac{\delta}{3}(9+7\delta)}.$$

Hence on putting $\mathbf{x} = (u_1, \dots, u_6, u_7a_7, \dots, u_7a_s)$, we have

$$\max|x_i| \ll (\max|a_i|)^{1+\frac{\delta}{3}(9+7\delta)},$$

so on taking

$$\gamma < \frac{3\delta}{3 + \delta(9 + 7\delta)}$$

we have

$$|F(\mathbf{x})| < (\max|a_i|)^{-\delta} < (\max|x_i|)^{-\gamma}.$$

Furthermore, if

$$\varepsilon < \frac{3\sigma - 2\delta(9 + 7\delta)}{3 + \delta(9 + 7\delta)}$$

then we have

$$|G(\mathbf{x})| < |u_7|^2(\max|a_i|)^{-\sigma} \ll (\max|a_i|)^{\frac{2\delta}{3}(9+7\delta)2-\sigma},$$

whence for $\max|a_i|$ sufficiently large

$$|G(\mathbf{x})| < (\max|x_i|)^{-\varepsilon}.$$

Thus we may take $\sigma_1 = \sigma_2 = 1.2 \times 10^{-2}$.

We note that throughout our arguments there is some freedom in the choice of the parameter δ, and we have generally chosen it so as to give roughly the same permissible values for σ_1 and σ_2. If so desired, one can alter δ in favor of one exponent or the other and in fact obtain a region of permissible values similar in shape to (but smaller than) the region in Theorem 1(d). We do not pursue this refinement here.

3. The Davenport-Heilbronn Method

We now set up a two-dimensional version of the Davenport-Heilbronn method which we will use to prove Theorem 1. We may assume (after rearranging variables) that the the first m of the μ_i are zero, that the last n of the λ_i are zero, and that the remaining $h = s - m - n$ indices have both λ_i and μ_i nonzero. Then when $s \geq 13$ we have by conditions (a) and (b) of Theorem 1 that

$$0 \leq m \leq 5, \quad 0 \leq n \leq 4, \quad \text{and} \quad h \geq 4. \tag{3.1}$$

Furthermore, we may suppose that λ_I/λ_J and μ_I/μ_J are algebraic irrationals, where

$$I = m + h - 2, \quad J = m + h - 1, \quad \text{and} \quad K = m + h.$$

Let ε be a small positive number, and choose $\eta > 0$ sufficiently small in terms of ε. Take P to be a large positive number, put $R = P^\eta$, and let

$$\mathcal{A}(P, R) = \{n \in [1, P] \cap \mathbb{Z} : p|n,\ p \text{ prime} \Rightarrow p \leq R\}.$$

Write $\boldsymbol{\alpha} = (\alpha, \beta)$, and define generating functions

$$F_i(\boldsymbol{\alpha}) = \sum_{1 \le x \le P} e(\lambda_i \alpha x^3 + \mu_i \beta x^2) \tag{3.2}$$

and

$$f_i(\boldsymbol{\alpha}) = \sum_{x \in \mathcal{A}(P,R)} e(\lambda_i \alpha x^3 + \mu_i \beta x^2). \tag{3.3}$$

It will also be convenient to write

$$g_i(\alpha) = f_i(\alpha, 0) \quad \text{and} \quad H_i(\beta) = F_i(0, \beta).$$

According to Davenport [10], for every integer r there exists a real-valued even kernel function K of one real variable such that

$$K(\alpha) \ll \min(1, |\alpha|^{-r}) \tag{3.4}$$

and

$$\int_{-\infty}^{\infty} e(\alpha t) K(\alpha) d\alpha \begin{cases} = 0, & \text{if } |t| \ge 1, \\ \in [0,1], & \text{if } |t| \le 1, \\ = 1, & \text{if } |t| \le \frac{1}{3}. \end{cases} \tag{3.5}$$

We set

$$\mathcal{K}(\boldsymbol{\alpha}) = K(\alpha P^{-\sigma_1}) K(\beta P^{-\sigma_2}).$$

Now let $N(P)$ be the number of solutions of (1.1) with

$$x_i \in \mathcal{A}(P, R) \qquad (i = 1, \dots, m + h - 3)$$

and

$$1 \le x_i \le P \qquad (i = m + h - 2, \dots, s).$$

By a familiar argument, $N(P)$ is bounded below by $P^{-\sigma_1 - \sigma_2} R(P)$, where

$$R(P) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}(\boldsymbol{\alpha}) \mathcal{G}(\boldsymbol{\alpha}) \mathcal{H}(\boldsymbol{\alpha}) \mathcal{K}(\boldsymbol{\alpha}) \, d\boldsymbol{\alpha}, \tag{3.6}$$

$$\mathcal{F}(\boldsymbol{\alpha}) = \prod_{i=1}^{m+h-3} f_i(\boldsymbol{\alpha}), \quad \mathcal{H}(\boldsymbol{\alpha}) = \prod_{i=m+h-2}^{m+h} F_i(\boldsymbol{\alpha}), \quad \text{and} \quad \mathcal{G}(\boldsymbol{\alpha}) = \prod_{i=m+h+1}^{s} F_i(\boldsymbol{\alpha}).$$

We dissect the plane into three main regions, imitating the standard dissection of the real line used in the treatment of a single inequality. The trivial region is defined by

$$\mathfrak{t} = \{\boldsymbol{\alpha} : |\alpha| > P^{\sigma_1 + \varepsilon} \text{ or } |\beta| > P^{\sigma_2 + \varepsilon}\}, \tag{3.7}$$

the major arc by

$$\mathfrak{M} = \{\boldsymbol{\alpha} : |\alpha| \le P^{-9/4} \text{ and } |\beta| \le P^{-5/4}\}, \tag{3.8}$$

and the minor arcs by

$$\mathfrak{m} = \mathbb{R}^2 \setminus (\mathfrak{t} \cup \mathfrak{M}). \tag{3.9}$$

Our plan is to show that $R(P) \gg P^{s-5}$, with the main contribution coming from the major arc. For r sufficiently large in terms of ε, it follows easily from (3.4) and (3.7) that the contribution to $R(P)$ from the trivial region is $o(P^{s-5})$. In the next section, we consider a finer dissection of the minor arcs which allows us to show that their contribution to $R(P)$ is also $o(P^{s-5})$, provided that σ_1 and σ_2 are confined to the region specified in Theorem 1. Finally, in Section 5, we apply standard methods to deal with the major arc.

4. The Minor Arcs

We begin by bounding the integral (3.6) in terms of others having somewhat more standard forms. We start by choosing a finite covering of $\mathfrak{m}$ by unit squares of the form $[c, c+1] \times [d, d+1]$. For $\mathfrak{n} \subset \mathfrak{m}$, let $\mathcal{U}_\mathfrak{n}$ denote the square for which the integral

$$\iint_{\mathfrak{n}\cap\mathcal{U}_\mathfrak{n}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha}$$

is maximal, and write $\mathfrak{n}^* = \mathfrak{n} \cap \mathcal{U}_\mathfrak{n}$. Then for $r > 1$ it follows from (3.4) that

$$\iint_{\mathfrak{n}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})\mathcal{K}(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \ll P^{\sigma_1+\sigma_2} \iint_{\mathfrak{n}^*} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha}. \tag{4.1}$$

Furthermore, by arguing as in the proof of Lemma 7.3 of Wooley [18], we see that

$$\iint_{\mathfrak{n}^*} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \ll \iint_{\mathfrak{n}^*} |f_i(\boldsymbol{\alpha})|^{h-3} |g_j(\alpha)|^m |H_k(\beta)|^n\, d\boldsymbol{\alpha} \tag{4.2}$$

for some i, j, and k (depending on $\mathfrak{n}$) satisfying

$$m+1 \le i \le m+h, \quad 1 \le j \le m, \quad \text{and} \quad m+h+1 \le k \le s.$$

In the course of an argument in which $\mathfrak{n}$ is fixed, we will employ the abbreviations

$$f = |f_i(\boldsymbol{\alpha})|, \quad g = |g_j(\alpha)|, \quad \text{and} \quad H = |H_k(\beta)|.$$

Finally, on recalling (3.1) and again mimicking the arguments of [18], we obtain

$$f^{h-3} g^m H^n \ll P^{s-13} \left(f^{10} + f^u H^{10-u} + g^u H^{10-u} + f^{10-u} g^u \right) \tag{4.3}$$

whenever $5 \le u \le 6$. For convenience, we introduce the notation

$$Q = P^{s-13+\sigma_1+\sigma_2}. \tag{4.4}$$

We are now in a position to make use of certain mean value estimates developed in Wooley [18], [20]. Those which we need are recorded for reference in the following lemma.

Lemma 4.1. *Suppose that*

$$m+1 \le i \le m+h, \quad 1 \le j \le m, \quad \textit{and} \quad m+h+1 \le k \le s.$$

Then for any unit square $\mathcal{U} = [c, c+1] \times [d, d+1]$, *we have*

(i) $\iint_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^{10}\, d\boldsymbol{\alpha} \ll P^{17/3+\varepsilon}$,

(ii) $\iint_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^6 |H_k(\beta)|^4\, d\boldsymbol{\alpha} \ll P^{21/4+\varepsilon}$,

(iii) $\iint_{\mathcal{U}} |g_j(\alpha)|^6 |H_k(\beta)|^4\, d\boldsymbol{\alpha} \ll P^{21/4+\varepsilon}$,

(iv) $\iint_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^4 |g_j(\alpha)|^6\, d\boldsymbol{\alpha} \ll P^{21/4+\varepsilon}$,

(v) $\iint_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^{14}\, d\boldsymbol{\alpha} \ll P^9$,

(vi) $\iint_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^8 \, |H_k(\beta)|^5 \, d\boldsymbol{\alpha} \ll P^8$,

(vii) $\iint_{\mathcal{U}} |g_j(\alpha)|^8 \, |H_k(\beta)|^5 \, d\boldsymbol{\alpha} \ll P^8$,

(viii) $\iint_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^6 \, |g_j(\alpha)|^8 \, d\boldsymbol{\alpha} \ll P^9$.

Proof. Part (i) follows from Theorem 2 of Wooley [20] on considering the underlying Diophantine equations and making a change of variables. Parts (iii), (v), and (vii) follow from the corresponding parts of Lemmata 7.2, 9.1, and 9.4 of Wooley [18] on making a change of variables and noting that the additional restrictions imposed on the variable ranges in that paper can be removed without affecting the arguments. For the remaining parts, we use the idea of the proof of Lemma 9.1(i) of [18] in a manner typified by (ii): Write

$$s_m(\mathbf{x}, \mathbf{y}) = (x_1^m - y_1^m) + (x_2^m - y_2^m) + (x_3^m - y_3^m)$$

and

$$H(\beta) = \sum_{1 \le x \le P} e(\beta x^2).$$

Then on making the change of variables $\alpha' = \lambda_i \alpha$ and $\beta' = \mu_k \beta$ we have

$$\iint_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^6 \, |H_k(\beta)|^4 \, d\boldsymbol{\alpha} \ll \iint_{\mathcal{U}'} \sum_{\mathbf{x},\mathbf{y}} e\left(s_3(\mathbf{x}, \mathbf{y})\alpha + \frac{\mu_i}{\mu_k} s_2(\mathbf{x}, \mathbf{y})\beta \right) |H(\beta)|^4 \, d\alpha \, d\beta,$$

where the summation is over $\mathbf{x}$ and $\mathbf{y}$ with $x_i, y_i \in \mathcal{A}(P, R)$ and where $\mathcal{U}' = [m_3, n_3] \times [m_2, n_2]$ for some integers m_j and n_j with $n_j - m_j \ll 1$. If we now let

$$c(\mathbf{x}, \mathbf{y}) = e\left(\frac{\mu_i}{\mu_k} s_2(\mathbf{x}, \mathbf{y})\beta \right),$$

then since $c(\mathbf{x}, \mathbf{y})$ is unimodular we obtain

$$\begin{aligned} \iint_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^6 \, |H_k(\beta)|^4 \, d\boldsymbol{\alpha} &\ll \int_{m_2}^{n_2} \left(\sum_{\mathbf{x},\mathbf{y}} c(\mathbf{x}, \mathbf{y}) \int_{m_3}^{n_3} e(s_3(\mathbf{x}, \mathbf{y})\alpha) d\alpha \right) |H(\beta)|^4 \, d\beta \\ &\ll P^{13/4+\varepsilon} \int_0^1 |H(\beta)|^4 \, d\beta \ll P^{21/4+\varepsilon} \end{aligned}$$

on using Theorem 4.4 of Vaughan [16] and considering the underlying Diophantine equations. □

Lemma 4.1 allows us to handle regions of $\mathfrak{m}$ on which $\mathcal{H}$ is suitably bounded. Fortunately, when F_I, F_J, or F_K is large, we also obtain a great deal of information from a theorem of Baker [2], a special case of which is recorded below.

Lemma 4.2. *Let $P > P_0(\varepsilon)$ and $A > P^{3/4+\varepsilon}$. If $|F_i(\boldsymbol{\alpha})| \ge A$ for some $i = I, J$, or K, then there exists a natural number $q < P^{3+\varepsilon}A^{-3}$ and integers a and b with $(q, a, b) = 1$ such that $|\lambda_i \alpha q - a| < P^{\varepsilon} A^{-3}$ and $|\mu_i \beta q - b| < P^{1+\varepsilon} A^{-3}$.*

Proof. This is Theorem 5.1 of [2] with $T = P^{3/4+\varepsilon}$, $M = 1$, and $k = 3$. □

Lemma 4.2 suggests further dissecting $\mathfrak{m}$ according to the behavior of F_I, F_J, and F_K. Thus we start by defining

$$\mathfrak{e} = \{\boldsymbol{\alpha} \in \mathfrak{m} : |F_i(\boldsymbol{\alpha})| \le P^{3/4+\varepsilon} \text{ for } i = I, J, K\}.$$

Now let

$$\mathfrak{f}(I) = \{\boldsymbol{\alpha} \in \mathfrak{m} : |F_I(\boldsymbol{\alpha})| > P^{3/4+\varepsilon},\ \max(|F_J(\boldsymbol{\alpha})|, |F_K(\boldsymbol{\alpha})|) \le P^{3/4+\varepsilon}\},$$

define $\mathfrak{f}(J)$ and $\mathfrak{f}(K)$ likewise, and put

$$\mathfrak{f} = \mathfrak{f}(I) \cup \mathfrak{f}(J) \cup \mathfrak{f}(K).$$

Similarly, let

$$\mathfrak{g}(I) = \{\boldsymbol{\alpha} \in \mathfrak{m} : |F_I(\boldsymbol{\alpha})| \le P^{3/4+\varepsilon},\ \min(|F_J(\boldsymbol{\alpha})|, |F_K(\boldsymbol{\alpha})|) > P^{3/4+\varepsilon}\},$$

define $\mathfrak{g}(J)$ and $\mathfrak{g}(K)$ likewise, and put

$$\mathfrak{g} = \mathfrak{g}(I) \cup \mathfrak{g}(J) \cup \mathfrak{g}(K).$$

Finally, define

$$\mathfrak{h} = \{\boldsymbol{\alpha} \in \mathfrak{m} : |F_i(\boldsymbol{\alpha})| > P^{3/4+\varepsilon} \text{ for } i = I, J, K\}.$$

The set $\mathfrak{e}$ can be handled quite easily. Using (4.1)–(4.4) and Lemma 4.1, we obtain

$$\begin{aligned}\iint_{\mathfrak{e}} |\mathcal{FGHK}|\, d\boldsymbol{\alpha} &\ll Q\left(P^{3/4+\varepsilon}\right)^3 \iint_{\mathcal{U}_{\mathfrak{e}}} \left(f^{10} + f^6H^4 + g^6H^4 + f^4g^6\right) d\boldsymbol{\alpha} \\ &\ll P^{s-13+\sigma_1+\sigma_2+9/4+3\varepsilon}\left(P^{17/3+\varepsilon} + P^{21/4+\varepsilon}\right) \\ &= o(P^{s-5}),\end{aligned}$$

provided that $\sigma_1 + \sigma_2 < 1/12$, since ε can be chosen arbitrarily small.

The rational approximations provided by Lemma 4.2 allow us to incorporate major arc techniques along the lines of Brüdern [5] and [6] in dealing with the sets $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{h}$. For this we require some additional definitions and lemmata. Define

$$\mathcal{M}(q, a, b) = \{\boldsymbol{\alpha} \in [0, 1]^2 : |q\alpha - a| < P^{-9/4} \text{ and } |q\beta - b| < P^{-5/4}\},$$

$$\mathcal{M} = \bigcup_{\substack{0 \le a, b \le q < P^{3/4} \\ (q,a,b)=1}} \mathcal{M}(q, a, b),$$

$$S(q, a, b) = \sum_{x=1}^{q} e\left(\frac{ax^3 + bx^2}{q}\right),$$

and

$$S_t^*(q) = \sum_{\substack{1 \le a, b \le q \\ (q,a,b)=1}} \left|q^{-1}S(q, a, b)\right|^t.$$

Lemma 4.3. *For $t > 6$, we have*

$$\sum_{q \le X} S_t^*(q) \ll 1.$$

Proof. Using Lemma 10.4 of Wooley [18] and proceeding as in Lemma 2.11 of Vaughan [17], one sees that $S_t^*(q)$ is multiplicative, so

$$\sum_{q\le X} S_t^*(q) \le \prod_p \left(1+\sum_{h=1}^{\infty} S_t^*(p^h)\right). \tag{4.5}$$

Whenever $(p^h,a,b)=1$, we have

$$S(p^h,a,b)\ll p^{2h/3+\varepsilon}$$

by Theorem 7.1 of Vaughan [17], but in the case that $(b,p)=1$ it follows from Theorem 1 of Loxton and Vaughan [11] that in fact

$$S(p^h,a,b)\ll p^{h/2}.$$

Thus we have

$$\begin{aligned} S_t^*(p^h) &= p^{-ht}\sum_{\substack{1\le a,b\le p^h\\(p,b)=1}} \Big|S(p^h,a,b)\Big|^t + p^{-ht}\sum_{\substack{1\le a,b\le p^h\\(p^h,a,b)=1\\(p,b)>1}} \Big|S(p^h,a,b)\Big|^t \\ &\ll p^{-ht}\Big(p^{2h+ht/2}+p^{2h-1+2ht/3+t\varepsilon}\Big), \end{aligned}$$

whence for $t>6$ we have

$$\sum_{h=1}^{\infty} S_t^*(p^h)\ll p^{-1-\delta}$$

for some $\delta>0$, and the result now follows immediately from (4.5). □

Write

$$F(\boldsymbol{\alpha})=\sum_{1\le x\le P} e(\alpha x^3+\beta x^2) \tag{4.6}$$

and

$$v(\boldsymbol{\alpha})=\int_0^P e(\alpha\gamma^3+\beta\gamma^2)\,d\gamma. \tag{4.7}$$

The following lemma provides a useful refinement of [18], Lemma 9.2.

Lemma 4.4. *For $t>6$, we have*

$$\iint_{\mathcal{M}} |F(\boldsymbol{\alpha})|^t\,d\boldsymbol{\alpha}\ll P^{t-5}.$$

Proof. When $\boldsymbol{\alpha}\in\mathcal{M}(q,a,b)$, write $\boldsymbol{\xi}=(\xi_3,\xi_2)=(\alpha-a/q,\beta-b/q)$ and

$$V(\boldsymbol{\alpha})=V(\boldsymbol{\alpha};q,a,b)=q^{-1}S(q,a,b)v(\boldsymbol{\xi}).$$

Then for $\boldsymbol{\alpha}\in\mathcal{M}(q,a,b)$ we have by Lemma 4.4 of Baker [2] that

$$F(\boldsymbol{\alpha})=V(\boldsymbol{\alpha})+O(q^{2/3+\varepsilon}).$$

Hence if $\mathcal{M}_1$ denotes the subset of $\mathcal{M}$ on which $|V(\boldsymbol{\alpha})|\le q^{2/3+\varepsilon}$, then we have

$$\iint_{\mathcal{M}_1} |F(\boldsymbol{\alpha})|^t\,d\boldsymbol{\alpha}\ll\sum_{q\le P^{3/4}}(q^{2/3+\varepsilon})^tP^{-7/2}\ll P^{t-5},$$

provided that $t > 9/2$. For $\boldsymbol{\alpha} \in \mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1$, we have $|V(\boldsymbol{\alpha})| > q^{2/3+\varepsilon}$ and hence $|F(\boldsymbol{\alpha})| \ll |V(\boldsymbol{\alpha})|$. Moreover, by Theorem 7.3 of Vaughan [17], we have

$$v(\boldsymbol{\xi}) \ll P(1 + P^2|\xi_2| + P^3|\xi_3|)^{-1/3} \ll P(1 + P^2|\xi_2|)^{-1/6}(1 + P^3|\xi_3|)^{-1/6},$$

and on combining this with Lemma 4.3 we obtain

$$\iint\limits_{\mathcal{M}} |V(\boldsymbol{\alpha})|^t d\boldsymbol{\alpha} \ll P^{t-5} \sum_{q \le P^{3/4}} S_t^*(q) \ll P^{t-5}$$

whenever $t > 6$. Thus we have

$$\iint\limits_{\mathcal{M}_2} |F(\boldsymbol{\alpha})|^t d\boldsymbol{\alpha} \ll P^{t-5}$$

for $t > 6$, and this completes the proof. □

The sets $\mathfrak{f}$ and $\mathfrak{g}$ can now be handled with little difficulty by applying major arc treatments to one or two of the variables. The key observation is that Baker's Theorem (Lemma 4.2) allows us to bound an integral of $|F_i(\boldsymbol{\alpha})|^t$ over $\mathfrak{f}(i)^*$ or $\mathfrak{g}(j)^*$ $(j \ne i)$ in terms of the integral considered in the previous lemma.

Using (4.1)–(4.4) as on $\mathfrak{e}$, we obtain for some $i = I, J$, or K that

$$\iint\limits_{\mathfrak{f}} |\mathcal{FGHK}|\, d\boldsymbol{\alpha} \ll Q\left(P^{3/4+\varepsilon}\right)^2 \iint\limits_{\mathfrak{f}(i)^*} |F_i| \left(f^{10} + f^6H^4 + g^6H^4 + f^4g^6\right) d\boldsymbol{\alpha}.$$

Then by Hölder's inequality we have

$$\iint\limits_{\mathfrak{f}(i)^*} |F_i|\, f^{10} d\boldsymbol{\alpha} \ll \left(\iint\limits_{\mathfrak{f}(i)^*} |F_i|^7\, d\boldsymbol{\alpha}\right)^{1/7} \left(\iint\limits_{\mathcal{U}_{\mathfrak{f}}} f^{10} d\boldsymbol{\alpha}\right)^{1/2} \left(\iint\limits_{\mathcal{U}_{\mathfrak{f}}} f^{14} d\boldsymbol{\alpha}\right)^{5/14},$$

and by Lemma 4.1 we have

$$\iint\limits_{\mathfrak{f}(i)^*} |F_i| \left(f^6H^4 + g^6H^4 + f^4g^6\right) d\boldsymbol{\alpha} \ll P^{25/4+\varepsilon}.$$

Hence on using Lemmata 4.1, 4.2, and 4.4, together with a change of variables, we find that

$$\iint\limits_{\mathfrak{f}} |\mathcal{FGHK}|\, d\boldsymbol{\alpha} \;\ll\; P^{s-13+\sigma_1+\sigma_2+3/2+2\varepsilon}\left(P^{19/3+\varepsilon} + P^{25/4+\varepsilon}\right) \;=\; o(P^{s-5}),$$

provided that $\sigma_1 + \sigma_2 < 1/6$.

Proceeding similarly but instead taking $u = 40/7$ in (4.3), we have for some $i \ne j$ among I, J, and K that

$$\begin{aligned} \iint\limits_{\mathfrak{g}} |\mathcal{FGHK}|\, d\boldsymbol{\alpha} \;&\ll\; QP^{3/4+\varepsilon} \iint\limits_{\mathfrak{g}(i)^*} |F_j|^2 \left(f^{10} + f^{\frac{40}{7}}H^{\frac{30}{7}} + g^{\frac{40}{7}}H^{\frac{30}{7}} + f^{\frac{30}{7}}g^{\frac{40}{7}}\right) d\boldsymbol{\alpha} \\ &\ll\; QP^{3/4+\varepsilon}\left(\iint\limits_{\mathfrak{g}(i)^*} |F_j|^7\, d\boldsymbol{\alpha}\right)^{2/7} \left(\mathcal{I}_1^{5/7} + \mathcal{I}_2^{5/7} + \mathcal{I}_3^{5/7} + \mathcal{I}_4^{5/7}\right), \end{aligned}$$

where

$$\mathcal{I}_1 = \iint_{\mathcal{U}_{\mathfrak{g}}} f^{14} d\boldsymbol{\alpha}, \qquad \mathcal{I}_2 = \iint_{\mathcal{U}_{\mathfrak{g}}} f^8 H^6 d\boldsymbol{\alpha},$$

$$\mathcal{I}_3 = \iint_{\mathcal{U}_{\mathfrak{g}}} g^8 H^6 d\boldsymbol{\alpha}, \qquad \mathcal{I}_4 = \iint_{\mathcal{U}_{\mathfrak{g}}} f^6 g^8 d\boldsymbol{\alpha}.$$

Thus we have

$$\iint_{\mathfrak{g}} |\mathcal{F}\mathcal{G}\mathcal{H}\mathcal{K}|\, d\boldsymbol{\alpha} \ \ll\ P^{s-13+\sigma_1+\sigma_2+3/4+\varepsilon}\left(P^7\right) \ =\ o(P^{s-5}),$$

provided that $\sigma_1 + \sigma_2 < 1/4$.

The set $\mathfrak{h}$ is somewhat more difficult to deal with, and it is here that we make use of the hypothesis that λ_I/λ_J and μ_I/μ_J are algebraic irrationals. We divide $\mathfrak{h}$ into two main components,

$$\mathfrak{h}_1 = \{\boldsymbol{\alpha} \in \mathfrak{h} : |\alpha| \geq P^{-9/4+\varepsilon}\} \quad \text{and} \quad \mathfrak{h}_2 = \mathfrak{h} \setminus \mathfrak{h}_1,$$

and we further subdivide $\mathfrak{h}_1^*$ and $\mathfrak{h}_2^*$ into $O\left((\log P)^2\right)$ dyadic subsets of the form

$$\mathfrak{h}_i(A, B) = \{\boldsymbol{\alpha} \in \mathfrak{h}_i^* : A < |F_I(\boldsymbol{\alpha})| \leq 2A,\ B < |F_J(\boldsymbol{\alpha})| \leq 2B\}.$$

We also write

$$\mathfrak{h}(A, B) = \mathfrak{h}_1(A, B) \cup \mathfrak{h}_2(A, B).$$

We now use a method introduced by Baker [1] to give an upper bound for the Lebesgue measure of $\mathfrak{h}_i(A, B)$. If $\boldsymbol{\alpha} \in \mathfrak{h}(A, B)$, then by Lemma 4.2 there exist natural numbers

$$q_I < P^{3+\varepsilon}A^{-3}, \quad q_J < P^{3+\varepsilon}B^{-3}, \quad q_K < P^{3/4} \tag{4.8}$$

and integers $a_i,\ b_i$ with $(q_i, a_i, b_i) = 1$ for $i = I, J, K$ such that

$$|\lambda_I \alpha q_I - a_I| < P^{\varepsilon}A^{-3}, \quad |\mu_I \beta q_I - b_I| < P^{1+\varepsilon}A^{-3}; \tag{4.9}$$

$$|\lambda_J \alpha q_J - a_J| < P^{\varepsilon}B^{-3}, \quad |\mu_J \beta q_J - b_J| < P^{1+\varepsilon}B^{-3}; \tag{4.10}$$

and

$$|\lambda_K \alpha q_K - a_K| < P^{-9/4}, \quad |\mu_K \beta q_K - b_K| < P^{-5/4}. \tag{4.11}$$

Notice that the inequalities (4.9) and (4.10) restrict $\boldsymbol{\alpha}$ to lie in a box $\mathcal{B}_I$ about the point $(a_I/(\lambda_I q_I), b_I/(\mu_I q_I))$ with

$$\text{meas}(\mathcal{B}_I) \ll q_I^{-2} P^{1+2\varepsilon} A^{-6} \tag{4.12}$$

and at the same time in a box $\mathcal{B}_J$ about $(a_J/(\lambda_J q_J), b_J/(\mu_J q_J))$ with

$$\text{meas}(\mathcal{B}_J) \ll q_J^{-2} P^{1+2\varepsilon} B^{-6}. \tag{4.13}$$

We first obtain a lower bound for $q_I q_J$. As in the proof of Lemma 11.1 of Vaughan [17], it follows from (4.9) and (4.10) that for $\boldsymbol{\alpha} \in \mathfrak{h}_1$ we have

$$\left| \frac{\lambda_I}{\lambda_J} - \frac{a_I q_J}{a_J q_I} \right| \ll P^{-9/4},$$

whereas by a well-known theorem of Roth [15] we have

$$\left|\frac{\lambda_I}{\lambda_J}-\frac{a_Iq_J}{a_Jq_I}\right| \gg \frac{1}{|a_Jq_I|^{2+\varepsilon}},$$

so that $|a_Jq_I| \gg P^{9/8-\varepsilon}$. Similarly, for $\boldsymbol{\alpha}\in\mathfrak{h}_2$ we have

$$\frac{1}{|b_Jq_I|^{2+\varepsilon}} \ll \left|\frac{\mu_I}{\mu_J}-\frac{b_Iq_J}{b_Jq_I}\right| \ll P^{-5/4},$$

and hence $|b_Jq_I| \gg P^{5/8-\varepsilon}$. Thus on using (4.9) and (4.10) and recalling the definitions (3.7)–(3.9) we obtain

$$q_Iq_J \gg \begin{cases} P^{9/8-\sigma_1-2\varepsilon}, & \text{if } \boldsymbol{\alpha}\in\mathfrak{h}_1 \\ P^{5/8-\sigma_2-2\varepsilon}, & \text{if } \boldsymbol{\alpha}\in\mathfrak{h}_2. \end{cases} \tag{4.14}$$

Next we observe that when $\boldsymbol{\alpha}\in\mathfrak{h}_1(A,B)$ there are $O(P^{9+3\varepsilon}A^{-9})$ corresponding triples (q_I,a_I,b_I) satisfying (4.8) and (4.9). Alternatively, there are $O(P^{9+3\varepsilon}B^{-9})$ triples (q_J,a_J,b_J) satisfying (4.8) and (4.10). On combining this with (4.12), (4.13), and (4.14) we obtain

$$\text{meas}(\mathfrak{h}_1(A,B)) \ll P^{71/8+\sigma_1+7\varepsilon}(AB)^{-15/2}. \tag{4.15}$$

When $\boldsymbol{\alpha}\in\mathfrak{h}_2(A,B)$ we necessarily have $a_I=a_J=0$ for P sufficiently large, so proceeding as above gives

$$\text{meas}(\mathfrak{h}_2(A,B)) \ll P^{51/8+\sigma_2+6\varepsilon}(AB)^{-6}. \tag{4.16}$$

On applying Hölder's inequality and Lemma 4.1 as before and writing $L=(\log P)^2$, we find that for some A and B

$$\begin{aligned}\iint_{\mathfrak{h}_1} |\mathcal{FGHK}|\,d\boldsymbol{\alpha} &\ll QL \iint_{\mathfrak{h}_1(A,B)} |F_IF_JF_K|\left(f^{10}+f^{\frac{40}{7}}H^{\frac{30}{7}}+g^{\frac{40}{7}}H^{\frac{30}{7}}+f^{\frac{30}{7}}g^{\frac{40}{7}}\right)d\boldsymbol{\alpha} \\ &\ll QP^{\varepsilon}\left(\iint_{\mathfrak{h}_1^*}|F_K|^{\frac{105}{16}}\,d\boldsymbol{\alpha}\right)^{\frac{16}{105}}\left(\iint_{\mathfrak{h}_1(A,B)}|F_IF_J|^{\frac{15}{2}}\,d\boldsymbol{\alpha}\right)^{\frac{2}{15}}\left(P^9\right)^{\frac{5}{7}}.\end{aligned}$$

Thus by (4.15) and Lemma 4.4 we have

$$\iint_{\mathfrak{h}_1} |\mathcal{FGHK}|\,d\boldsymbol{\alpha} \ll P^{s-13+\frac{45}{7}+\frac{5}{21}+\frac{2}{15}(\frac{71}{8})+\frac{17}{15}\sigma_1+\sigma_2+2\varepsilon} = o(P^{s-5}),$$

provided that $\frac{17}{15}\sigma_1+\sigma_2<\frac{3}{20}$.

Since $\mathfrak{h}_2$ is a thin strip along the β-axis, we save a factor of P^{σ_1} in the analysis leading to (4.1), but the treatment is otherwise similar to the above. On writing $Q'=P^{s-13+\sigma_2}$, we have

$$\begin{aligned}\iint_{\mathfrak{h}_2} |\mathcal{FGHK}|\,d\boldsymbol{\alpha} &\ll Q'L \iint_{\mathfrak{h}_2(A,B)} |F_IF_JF_K|\left(f^{10}+f^{\frac{40}{7}}H^{\frac{30}{7}}+g^{\frac{40}{7}}H^{\frac{30}{7}}+f^{\frac{30}{7}}g^{\frac{40}{7}}\right)d\boldsymbol{\alpha} \\ &\ll P^{s-13+\sigma_2+\varepsilon}\left(\iint_{\mathfrak{h}_2^*}|F_K|^{\frac{42}{5}}\,d\boldsymbol{\alpha}\right)^{\frac{5}{42}}\left(\iint_{\mathfrak{h}_2(A,B)}|F_IF_J|^{6}\,d\boldsymbol{\alpha}\right)^{\frac{1}{6}}\left(P^9\right)^{\frac{5}{7}},\end{aligned}$$

whence by (4.16) we obtain

$$\iint_{\mathfrak{h}_2} |\mathcal{FGHK}|\, d\boldsymbol{\alpha} \ll P^{s-13+\frac{45}{7}+\frac{17}{42}+\frac{1}{6}(\frac{51}{8})+\frac{7}{6}\sigma_2+2\varepsilon} = o(P^{s-5}),$$

provided that $\frac{7}{6}\sigma_2 < \frac{5}{48}$. It is easily seen that these last two inequalities are less restrictive than the one appearing in condition (d) of Theorem 1.

5. The Major Arc

As it stands, the major arc $\mathfrak{M}$ is too large to allow satisfactory approximation of the exponential sums $f_i(\boldsymbol{\alpha})$, so we must do some pruning. Specifically, let W be a parameter at our disposal, and let

$$\mathfrak{N} = \{\boldsymbol{\alpha} : |\alpha| \le WP^{-3} \text{ and } |\beta| \le WP^{-2}\}. \tag{5.1}$$

Then as in Lemma 9.2 of Wooley [18], we have for $t > 9$ that

$$\iint_{\mathfrak{M}\setminus\mathfrak{N}} |F_i(\boldsymbol{\alpha})|^t\, d\boldsymbol{\alpha} \ll W^{-\sigma} P^{t-5}$$

for $i = I, J, K$ and some $\sigma > 0$. Thus by using (4.3) and Lemma 4.1 as in the treatment of $\mathfrak{g}$ and $\mathfrak{h}$ in the previous section, we have for some $i = I, J$, or K that

$$\begin{aligned}\iint_{\mathfrak{M}\setminus\mathfrak{N}} |\mathcal{FGHK}|\, d\boldsymbol{\alpha} &\ll P^{s-13}\left(\iint_{\mathfrak{M}\setminus\mathfrak{N}} |F_i(\boldsymbol{\alpha})|^{21/2}\, d\boldsymbol{\alpha}\right)^{2/7} P^{45/7} \\ &\ll P^{s-5}W^{-\sigma'}.\end{aligned}$$

It remains to deal with the pruned major arc $\mathfrak{N}$. Let

$$v_i(\boldsymbol{\alpha}) = \int_0^P e(\lambda_i\alpha\gamma^3 + \mu_i\beta\gamma^2)\, d\gamma \tag{5.2}$$

and

$$w_i(\boldsymbol{\alpha}) = \int_R^P \rho\left(\frac{\log\gamma}{\log R}\right) e(\lambda_i\alpha\gamma^3 + \mu_i\beta\gamma^2)\, d\gamma, \tag{5.3}$$

where $\rho(x)$ is Dickman's function (see Vaughan [17], chapter 12). Then for $\boldsymbol{\alpha} \in \mathfrak{N}$, we obtain from Theorem 7.2 of [17] that

$$F_i(\boldsymbol{\alpha}) = v_i(\boldsymbol{\alpha}) + O(W)$$

and from Lemma 8.5 of [18] that

$$f_i(\boldsymbol{\alpha}) = w_i(\boldsymbol{\alpha}) + O(WP/\log P).$$

Now on taking $W = (\log P)^{1/4}$ it follows that

$$\iint_{\mathfrak{N}} \mathcal{FGHK}\, d\boldsymbol{\alpha} = \iint_{\mathfrak{N}} \left(\prod_{i=1}^{m+h-3} w_i(\boldsymbol{\alpha})\right)\left(\prod_{i=m+h-2}^{s} v_i(\boldsymbol{\alpha})\right)\mathcal{K}(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} + O(P^{s-5}W^{-1}).$$

Furthermore, we may extend the integration over all of $\mathbb{R}^2$, as the bounds for v_i and w_i contained in Lemma 8.6 of [18] imply that

$$\iint_{\mathbb{R}^2\setminus\mathfrak{N}} \left(\prod_{i=1}^{m+h-3} w_i(\boldsymbol{\alpha})\right)\left(\prod_{i=m+h-2}^{s} v_i(\boldsymbol{\alpha})\right)\mathcal{K}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \ll P^{s-5}W^{-1}.$$

Thus it remains to show that the singular integral

$$J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{m+h-3} w_i(\boldsymbol{\alpha})\right)\left(\prod_{i=m+h-2}^{s} v_i(\boldsymbol{\alpha})\right)\mathcal{K}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}$$

satisfies $J \gg P^{s-5}$. Multiplying out, we have

$$J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathcal{B}^*} T^*(\boldsymbol{\gamma})\,e(F(\boldsymbol{\gamma})\alpha + G(\boldsymbol{\gamma})\beta)\,K(\alpha P^{-\sigma_1})K(\beta P^{-\sigma_2})\,d\boldsymbol{\gamma}\,d\alpha\,d\beta,$$

where

$$\mathcal{B}^* = [R,P]^{m+h-3}\times[0,P]^{n+3}$$

and

$$T^*(\boldsymbol{\gamma}) = \prod_{i=1}^{m+h-3}\rho\left(\frac{\log\gamma_i}{\log R}\right).$$

On making the change of variables

$$\boldsymbol{\gamma}' = \boldsymbol{\gamma}P^{-1},\quad \alpha' = \alpha P^{-\sigma_1},\quad \beta' = \beta P^{-\sigma_2}$$

and applying Fubini's Theorem, we obtain

$$J = P^{s+\sigma_1+\sigma_2}\int_{\mathcal{B}} T(\boldsymbol{\gamma})\,\hat{K}(F(\boldsymbol{\gamma})P^{3+\sigma_1})\hat{K}(G(\boldsymbol{\gamma})P^{2+\sigma_2})\,d\boldsymbol{\gamma}, \tag{5.4}$$

where we have written

$$\mathcal{B} = P^{-1}\mathcal{B}^*, \qquad T(\boldsymbol{\gamma}) = T^*(P\boldsymbol{\gamma}),$$

and

$$\hat{K}(t) = \int_{-\infty}^{\infty} e(\alpha t)K(\alpha)d\alpha.$$

Now by condition (c) of Theorem 1 and the argument of Lemma 6.2 of Wooley [18], we can find a non-singular solution $\boldsymbol{\eta}$ to the equations $F = G = 0$ such that each η_i is non-zero. Then, after replacing λ_i by $-\lambda_i$ if necessary and using homogeneity, we may assume that $\boldsymbol{\eta} \in (0,1)^s$ and hence that $\boldsymbol{\eta}$ lies in the interior of $\mathcal{B}$ when P is sufficiently large. Suppose that $6\eta_j\eta_k(\lambda_j\mu_k\eta_j - \lambda_k\mu_j\eta_k) \neq 0$, and consider the map $\phi : \mathbb{R}^s \to \mathbb{R}^s$ defined by

$$\phi_j = F(\boldsymbol{\gamma}),\quad \phi_k = G(\boldsymbol{\gamma}),\quad \text{and}\quad \phi_i = \gamma_i \quad (i \neq j,k). \tag{5.5}$$

By the inverse function theorem, there exist neighborhoods U of $\boldsymbol{\eta}$ and V of $\phi(\boldsymbol{\eta})$ such that ϕ maps U injectively onto V, and we may assume that $U \subset \mathcal{B}$. Now by (3.5) and the nonnegativity of ρ, the integrand in (5.4) is nonnegative, so we may restrict the integration over $\boldsymbol{\gamma}$ to the set U. Then on writing $\mathbf{z} = \phi(\boldsymbol{\gamma})$, where ϕ is as in (5.5), we have by the change of variables theorem that

$$J \geq P^{s+\sigma_1+\sigma_2}\int_V T(\phi^{-1}(\mathbf{z}))\hat{K}(z_jP^{3+\sigma_1})\hat{K}(z_kP^{2+\sigma_2})\left|\frac{d\boldsymbol{\gamma}}{d\mathbf{z}}\right| d\mathbf{z}. \tag{5.6}$$

Since meas$(V) \gg 1$, the projection of V onto z_j contains the interval $[0, \frac{1}{3}P^{-3-\sigma_1}]$, and the projection of V onto z_k contains the interval $[0, \frac{1}{3}P^{-2-\sigma_2}]$, provided that P is sufficiently large. Hence on restricting the range of integration in (5.6) and using (3.5) again, we obtain

$$J \gg P^{s+\sigma_1+\sigma_2} \int_{\mathcal{S}} T(\phi^{-1}(\mathbf{z}))d\mathbf{z},$$

where meas$(\mathcal{S}) \gg P^{-5-\sigma_1-\sigma_2}$. Finally, on noting that $T(\boldsymbol{\gamma}) \gg \rho(1/\eta)^{m+h-3} \gg 1$ for $\boldsymbol{\gamma} \in \mathcal{B}$, we obtain $J \gg P^{s-5}$ as required. This completes the proof of Theorem 1.

6. A Discussion of the Conditions in Theorems 1 and 2

Here we discuss the possibility of weakening some of the conditions imposed on the forms F and G in Theorems 1 and 2. In view of the discussion of Wooley [18], §5, where it is shown that many conditions similar to ours are essentially best possible for the corresponding problem on equations, our observations will leave something to be desired. Nevertheless, we can show that at least some minimal conditions are necessary to ensure the solubility of (1.1).

For example, let

$$F(\mathbf{x}) = \lambda^3 x_1^3 - x_2^3 \quad \text{and} \quad G(\mathbf{x}) = \mu^2 x_3^2 - x_4^2,$$

where λ and μ are positive real algebraic of degree 3 and 2, respectively, such that λ^3 and μ^2 are irrational. For instance, we may take $\lambda = 1 + \sqrt[3]{2}$ and $\mu = 1 + \sqrt{2}$. Then it follows easily from Liouville's Theorem that, for sufficiently small $\tau > 0$, neither of the inequalities

$$|F(\mathbf{x})| < \tau, \quad |G(\mathbf{x})| < \tau$$

has a non-trivial solution in rational integers. Of course, this example is easily generalized to produce forms $F_1, \dots, F_t$ of degrees $k_1, \dots, k_t$ in $2t$ variables which do not take arbitrarily small values. Therefore, we must minimally require either $s \geq 5$ total variables or at least 3 variables explicit in one of the two forms.

More realistically, in light of [20], Theorem 1, one might hope to be able to prove Theorem 1 with $s = 13$ but conditions (a) and (b) weakened so that F and G need only have 7 and 5 variables explicit, respectively, rather than 9 and 8. The latter numbers arise from the inequalities (3.1), on which the analytic argument in Sections 2.3–2.5 depends, but one may attempt to reduce these in the manner of [18] and [20] by using Theorem 2. Unfortunately, there are some difficulties with this approach in our situation. If F has exactly 7 or 8 variables explicit, then we may apply Theorem 2 to solve (1.1), but we must settle for the inferior values of σ_1 and σ_2 allowed by condition (d)(i) of that theorem, and we forfeit our estimate for the density of solutions. Moreover, if G has exactly 7 variables explicit and F has at least 10 variables explicit, then neither Theorem 1 nor Theorem 2 applies with $s = 13$. To avoid this difficulty, we may hope to reduce the number of zero coefficients required by condition (d)(ii) of the latter from 7 to 6, and we saw in Section 2.2 that a conditional result of this type could be obtained using hypothetical results on small solutions of cubic inequalities in 7 variables.

As mentioned in Section 2.1, condition (b) of Theorem 2 can be eliminated from the stated version of the theorem, but some form of it is likely to be necessary for any desirable refinement of (d)(ii). If a quantitative version of the result of Margulis [12] on the Oppenheim conjecture were available, then we could reduce the 5 to 3 in condition (b) of our hypothetical version of Theorem 2, provided we assumed additionally that G is not a multiple of a form with integer coefficients. However, the methods of [12] do not seem to hold much promise for obtaining such a result.

We can also investigate the possibility of reducing the total number of variables required. Although Theorem 1 could conceivably hold with as few as 5 variables, it does not seem possible for an analytic argument of the flavor given in Sections 2.3–2.5 to be successful with fewer than 11 variables. In the "ideal" situation that the first four mean values in Lemma 4.1 were bounded by $P^{5+\varepsilon}$, a simplified version of our analysis would allow us to prove a version of the theorem for $s \geq 12$, possibly with a slightly different range of permissible values for σ_1 and σ_2.

Next we note that the existence of a non-trivial real solution to the equations $F = G = 0$ is a necessary condition for the system (1.1) to have infinitely many integer solutions. For, if the latter holds, then for arbitrary $\tau > 0$ we can obtain (by rescaling an integer solution $\mathbf{x}$ with $\max |x_i|$ sufficiently large) a real solution $\boldsymbol{\eta}(\tau) \in [-1,1]^s$ of the inequalities $|F| < \tau,\ |G| < \tau$ such that $|\eta_i| = 1$ for some i. But the set

$$\mathcal{S} = \{\boldsymbol{\eta} \in [-1,1]^s : |\eta_i| = 1 \text{ for some } i\}$$

is compact, whence its image in $\mathbb{R}^2$ under the continuous map ϕ defined by F and G is compact. Hence $\phi(\mathcal{S})$ must contain the limit point $(0,0)$, which shows that the equations $F = G = 0$ have a non-trivial real solution.

Now let p be a prime with $p \equiv 1 \pmod 3$, let c be a cubic nonresidue $(\mathrm{mod}\ p)$, and consider the forms

$$\begin{aligned} F(\mathbf{x}) &= \sqrt{2}x_1^3 + x_2^3 + \cdots + x_7^3 + (x_8^3 + cx_9^3) + p(x_{10}^3 + cx_{11}^3) + p^2(x_{12}^3 + cx_{13}^3), \\ G(\mathbf{x}) &= \sqrt{2}x_1^2 + x_2^2 + \cdots + x_7^2 + x_8^2. \end{aligned}$$

It is easily checked that F and G satisfy all the conditions of Theorem 1, except that all real solutions to the simultaneous equations $F = G = 0$ are singular. Moreover, the discussion of example (5.1) in Wooley [18] shows that the simultaneous inequalities

$$|F(\mathbf{x})| < 1, \quad |G(\mathbf{x})| < 1$$

have no nontrivial integer solutions. Therefore, condition (c) of Theorem 1 cannot be weakened.

We conclude with some remarks on the assumption regarding algebraic irrational coefficient ratios in Theorem 1. First of all, if neither F nor G is a multiple of a form with integer coefficients and all the coefficients of F and G are nonzero, then it is easy to see that there is a pair of indices i and j such that both λ_i/λ_j and μ_i/μ_j are irrational. Next, if exactly one of the forms is a multiple of an integral form and this form has no zero coefficients, then we can solve the problem by obtaining a lower bound for the integral

$$R_1(P) = \int_{-\infty}^{\infty} \int_0^1 \mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})K(\alpha P^{-\sigma_1})\, d\beta\, d\alpha$$

or

$$R_2(P) = \int_{-\infty}^{\infty} \int_0^1 \mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})K(\beta P^{-\sigma_2})\, d\alpha\, d\beta,$$

as the case may be, using a simplified version of our analysis, along with techniques from the one-dimensional Hardy-Littlewood and Davenport-Heilbronn methods. If F and G are both multiples of integral forms, then we may simply apply the argument of Wooley [20] to deduce Theorem 1. Thus in particular we observe that if all the coefficients of F and G are algebraic and nonzero, then no irrationality assumption on the coefficients is needed.

The algebraicity assumption allows us to use Roth's Theorem in Section 2.4 to obtain the lower bounds (4.14), which are critical to our analysis of the sets $\mathfrak{h}_i(A, B)$. The preferred approach to (4.14) would involve restricting P in terms of the denominators of simultaneous rational approximations $\lambda_I/\lambda_J \sim a/q$ and $\mu_I/\mu_J \sim b/q$ and then combining these approximations with (4.9) and (4.10), in analogy with the proof of [17], Lemma 11.1. However, a difficulty arises from the possibility that (a, q) or (b, q) may be large, even though we can ensure that $(q, a, b) = 1$. It transpires that in this problematic case we can reduce the task to one of obtaining small solutions to "mixed" systems of the form

$$|F(\mathbf{x})| < (\max|x_i|)^{-\sigma_1}, \quad \sum_{i=1}^{s} b_i x_i^2 = 0$$

or

$$|G(\mathbf{x})| < (\max|x_i|)^{-\sigma_2}, \quad \sum_{i=1}^{s} a_i x_i^3 = 0,$$

where the a_i and b_i are integers. Under suitable conditions, the number of solutions to these systems can be estimated as described above, using integrals like $R_1(P)$ and $R_2(P)$. However, in order to obtain bounds for the solutions in terms of the coefficients of the forms, we must now keep track of constants which were previously left implicit, and this would seem to require additional information regarding the nature of a real solution to the corresponding system of equations.

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