THE DENSITY OF RATIONAL LINES ON CUBIC HYPERSURFACES

SCOTT T. PARSELL

ABSTRACT. We provide a lower bound for the density of rational lines on the hypersurface defined by an additive cubic equation in at least 57 variables. In the process, we obtain a result on the paucity of non-trivial solutions to an associated system of Diophantine equations.

1. INTRODUCTION

The existence of linear spaces on algebraic varieties was first investigated by Brauer [4] and Birch [3] in the middle part of the 20th century, but the analysis of the density of such spaces has begun only very recently. The author [10] has shown that if $c_1, \ldots, c_s$ are non-zero integers then the hypersurface defined by the additive equation

$$c_1 z_1^k + \cdots + c_s z_s^k = 0$$

contains the expected density of rational lines, provided that

$$s \geq (\tfrac{14}{3} + o(1))k^2 \log k$$

and that the obvious local solubility requirements are met. The purpose of the present note is to obtain an explicit upper bound for the number of variables required in the case $k = 3$.

Given an affine line $\ell : \mathbf{x}t + \mathbf{y}$, we define its "height" by $h(\ell) = \max(|x_i|, |y_i|)$. Further, when $c_1, \ldots, c_s$ are non-zero integers, write $L_s(P)$ for the number of distinct rational lines ℓ having $h(\ell) \leq P$ and lying on the hypersurface defined by

$$c_1 z_1^3 + \cdots + c_s z_s^3 = 0.$$

Our main result is the following.

Theorem 1. *Suppose that $s \geq 57$. Then one has*

$$L_s(P) \gg P^{2s-12}$$

for P sufficiently large.

For comparison, we note that Wooley [19] has demonstrated the existence of rational lines on arbitrary cubic hypersurfaces in at least 37 variables, whereas we require 57 variables in Theorem 1. In the additive situation we are considering, the existence of lines follows immediately from the theory of a single additive cubic equation (see R. Baker [2]), provided that $s \geq 14$, and hence the significance of our result lies in the density estimate.

The existence of these "trivial" lines when $s \geq 14$ is in fact key to our analysis, for they give rise to non-singular integer solutions of the system

$$c_1 x_1^{3-j} y_1^j + \cdots + c_s x_s^{3-j} y_s^j = 0 \quad (0 \leq j \leq 3) \tag{1.1}$$

and hence allow us to avoid imposing explicit local solubility hypotheses in Theorem 1. Unfortunately, the solutions arising in this way are singular for larger values of k and are therefore of no use in the analysis of the more general problem considered in [10]. Local conditions also may present an obstacle to demonstrating the expected density of higher-dimensional linear spaces on a cubic hypersurface in a reasonable number of variables. While the results of Schmidt [11], [12] could be applied to the analogues of (1.1), the number of variables required may in general be quite large.

We prove Theorem 1 by applying the Hardy-Littlewood method to count the number of solutions of (1.1) lying in a given box. In order to obtain estimates for mean values of the underlying exponential sums, we will be required to investigate certain auxiliary systems of equations. For example, let $S(P)$ denote the number of solutions of the system

$$
\begin{aligned}
x_1^3 + x_2^3 + x_3^3 &= x_4^3 + x_5^3 + x_6^3 \\
x_1^2 y_1 + x_2^2 y_2 + x_3^2 y_3 &= x_4^2 y_4 + x_5^2 y_5 + x_6^2 y_6 \\
x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2 &= x_4 y_4^2 + x_5 y_5^2 + x_6 y_6^2 \\
y_1^3 + y_2^3 + y_3^3 &= y_4^3 + y_5^3 + y_6^3
\end{aligned}
\tag{1.2}
$$

with $x_i, y_i \in [1, P] \cap \mathbb{Z}$. Further, write $T(P)$ for the number of "trivial" solutions, in which

$$
(x_4, x_5, x_6) = \sigma(x_1, x_2, x_3) \quad \text{and} \quad (y_4, y_5, y_6) = \sigma(y_1, y_2, y_3)
$$

for some permutation $\sigma \in S_3$. The following "paucity" result shows that almost all solutions counted by $S(P)$ are of this diagonal type.

Theorem 2. *For every $\varepsilon > 0$, one has*

$$
S(P) - T(P) \ll_\varepsilon P^{6 - \frac{11}{192} + \varepsilon}.
$$

Clearly, one has $T(P) = 6P^6 + O(P^4)$, and it follows that $S(P) \sim 6P^6$. In Section 3, we interpret this diagonal behavior as giving a best-possible estimate for the 6th moment of the exponential sum

$$
F(\boldsymbol{\alpha}) = \sum_{1 \le x, y \le P} e(\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 x y^2 + \alpha_3 y^3)
$$

and then use the iterative method developed in [10] to obtain non-trivial estimates for higher moments. After deriving some simple Weyl estimates in Section 4, we are able to complete the proof of Theorem 1 in Section 5 by using the Hardy-Littlewood method as in [10], §10.

Our methods apply equally well to a generalization of Waring's problem in which we seek to represent a polynomial of degree k as a sum of kth powers of linear polynomials. This application was discussed in detail in [10], where the function $G_1^*(k)$ was introduced to denote the number of kth powers required to represent all polynomials whose coefficients are sufficiently large and compatible with all local solubility considerations. The bound $G_1^*(3) \le 55$ is immediate on combining the analysis of [10], §9, with that of the present paper, so we feel justified in omitting the details.

2. The Paucity Problem

Our goal in this section is to establish Theorem 2. Before proceeding with the proof, we record for reference some of the key estimates we will use. The first of these is implicit in the work of Hooley [7] on sums of four cubes.

Lemma 2.1. *Let n be a non-zero integer, and let $R(P)$ denote the number of integral solutions of the equation*

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = n$$

with $|x_i| \le P$. Then one has $R(P) \ll P^{11/6+\varepsilon}$.

Proof. Clearly, we may focus attention on solutions in which at least two of the x_i are non-zero. For any such solution $\mathbf{x}$ counted by $R(P)$, we can find i and j such that x_i and x_j have the same parity and are not both zero. Now if $x_i + x_j = 0$ and x_k and x_l are the remaining two variables, then since $n \ne 0$ we must have $x_k + x_l \ne 0$, and if x_k and x_l do not have the same parity, then one of them has the same parity as x_i and x_j. Thus, after relabeling variables, we may assume that $x_1 \equiv x_2 \pmod{2}$ and $x_1 \ne -x_2$. This allows us to write $x_1 = r + s$ and $x_2 = r - s$, where r and s are integers with $r \ne 0$, and hence to consider solutions of the equation

$$2r(r^2 + 3s^2) = n - z^3 - w^3.$$

The argument is now identical to that of Hooley [7], the condition $r \ne 0$ being essential to the consideration of congruences modulo divisors of r. The only change is that the upper bound of $n^{1/3}$ for the moduli of r, z, and w is replaced throughout by P, and the sieving parameter ξ is now chosen to be $P^{1/6}$. □

We also make use of some recent work of Heath-Brown [6] on sums of two cubes.

Lemma 2.2. *Let $U(P)$ denote the number of integral solutions of the equation*

$$x_1^3 + x_2^3 = x_3^3 + x_4^3$$

with $|x_i| \le P$ and $x_1 + x_2 \ne x_3 + x_4$. Then one has $U(P) \ll P^{4/3+\varepsilon}$.

Proof. This is a special case of Heath-Brown [6], Theorem 1. □

We remark that Hooley [8], using the Riemann hypothesis for varieties over finite fields, obtained a result of the above shape with the exponent $4/3$ replaced by $5/3$. Wooley [18] later devised an elementary proof of this result, and his ideas play a key role in Heath-Brown's argument.

Finally, we recall a result on binary quadratic forms dating back to Estermann [5].

Lemma 2.3. *Let a, b, and c be non-zero integers, and let $Q(P)$ denote the number of integral solutions of the equation*

$$ax^2 + by^2 = c$$

with $1 \le x, y \le P$. Then one has $Q(P) \ll |abcP|^\varepsilon$.

Proof. See (for example) Vaughan and Wooley [15], Lemma 3.5. □

We are now ready to embark on the proof of Theorem 2. On writing $h = x_1 - x_4$ and $g = y_1 - y_4$ and relabeling variables in (1.2), we see that $S(P)$ is the number of solutions of the system of equations

$$
\begin{aligned}
h(3x(x + h) + h^2) &= u_1^3 + u_2^3 - u_3^3 - u_4^3 \\
(2hx + h^2)y + g(x + h)^2 &= u_1^2 v_1 + u_2^2 v_2 - u_3^2 v_3 - u_4^2 v_4 \\
(2gy + g^2)x + h(y + g)^2 &= u_1 v_1^2 + u_2 v_2^2 - u_3 v_3^2 - u_4 v_4^2 \\
g(3y(y + g) + g^2) &= v_1^3 + v_2^3 - v_3^3 - v_4^3
\end{aligned}
\tag{2.1}
$$

with

$$
1 \le x, y, u_i, v_i \le P \quad \text{and} \quad |h|, |g| < P.
\tag{2.2}
$$

We shall estimate $N(P) = S(P) - T(P)$ by dividing into several cases.

(i) Let N_1 denote the number of solutions counted by $N(P)$ for which $h = g = 0$, and consider a solution $x, y, \mathbf{u}, \mathbf{v}$ counted by N_1. Then one has

$$
(u_1, u_2, v_1, v_2) \ne (u_3, u_4, v_3, v_4) \quad \text{and} \quad (u_1, u_2, v_1, v_2) \ne (u_4, u_3, v_4, v_3),
$$

since otherwise the solution would be counted by $T(P)$. If we have $(u_1, u_2) = (u_3, u_4)$ and $(v_1, v_2) = (v_4, v_3)$, then the second equation in (2.1) implies that either $u_1 = u_2$ or $v_1 = v_2$, whence the number of choices for $\mathbf{u}$ and $\mathbf{v}$ is $O(P^3)$. Trivially, there are $O(P^2)$ choices for x and y, so the total number of solutions is $O(P^5)$, and the same analysis applies if $(u_1, u_2) = (u_4, u_3)$ and $(v_1, v_2) = (v_3, v_4)$. Otherwise, since u_i and v_i are positive, it follows that either $u_1 + u_2 \ne u_3 + u_4$ or $v_1 + v_2 \ne v_3 + v_4$, so Lemma 2.2 may be applied to estimate the number of choices for $\mathbf{u}$ or $\mathbf{v}$ (or possibly both). On combining this with Hua's Lemma, one sees that $N_1 \ll P^{16/3+\varepsilon}$.

(ii) Let N_2 denote the number of solutions counted by $N(P)$ for which exactly one of h or g is zero. Suppose first that $h = 0$ and $g \ne 0$. Then by Hua's Lemma one has $O(P^{2+\varepsilon})$ choices for $\mathbf{u}$, and by a trivial estimate there are $O(P^2)$ choices for g and y. Now for fixed non-zero g and y, we may apply Lemma 2.1 to deduce that there are $O(P^{11/6+\varepsilon})$ choices of $\mathbf{v}$ satisfying the fourth equation of (2.1). Finally, since $g \ne 0$, the second equation is a non-trivial polynomial in x and hence determines x to $O(1)$. By following a symmetric argument in the case where $g = 0$ and $h \ne 0$, we find that $N_2 \ll P^{35/6+\varepsilon}$.

(iii) Write $d = (h, g)$, let β be a parameter at our disposal, and let N_3 denote the number of solutions counted by $N(P)$ for which $hg \ne 0$ and $|hg/d| \le P^{1+\beta}$. In this case, there are $O(P^{1+\beta})$ choices for the integer hg/d, of which d, h/d and g/d are all divisors. Thus by a standard estimate for the divisor function, we see that there are $O(P^{1+\beta+\varepsilon})$ choices for h and g. Trivially, there are $O(P)$ choices for x, and then by Lemma 2.1 we have $O(P^{11/6+\varepsilon})$ choices for $\mathbf{u}$. Now by taking a linear combination of the equations (2.1), with respective weights $g^3, -3g^2h, 3gh^2$, and $-h^3$, we find that any solution $x, y, g, h, \mathbf{u}, \mathbf{v}$ satisfies

$$
(gu_1 - hv_1)^3 + (gu_2 - hv_2)^3 = (gu_3 - hv_3)^3 + (gu_4 - hv_4)^3,
\tag{2.3}
$$

and by applying Hua [9], Theorem 4, to the underlying mean value we find that, for fixed h, g, and $\mathbf{u}$, there are $O(P^{2+\varepsilon})$ choices for $\mathbf{v}$. Finally, y is determined to $O(1)$ by a polynomial, whence $N_3 \ll P^{35/6+\beta+\varepsilon}$.

(iv) For $i = 1, \ldots, 4$, write $X_i = gu_i - hv_i$, and let N_4 denote the number of solutions counted by $N(P)$ for which $X_1 + X_2 = X_3 + X_4$ and $hg \neq 0$. The former condition, when combined with (2.3), implies that either $X_1 = X_3$, $X_2 = X_3$, or $X_1 = -X_2$. We may suppose that $X_1 = X_3$ and $X_2 = X_4$, so that

$$g(u_1 - u_3) = h(v_1 - v_3) \quad \text{and} \quad g(u_2 - u_4) = h(v_2 - v_4), \qquad (2.4)$$

the argument in the remaining two cases being identical. For convenience we write

$$A = u_1 - u_3, \quad B = u_2 - u_4, \quad C = v_1 - v_3, \quad \text{and} \quad D = v_2 - v_4. \qquad (2.5)$$

Since h and g are non-zero, the first equation in (2.1) implies that either A or B is non-zero, and the fourth equation implies that either C or D is non-zero. Suppose that $C \neq 0$ and $D = 0$. We first choose $u_2 = u_4$ and $v_2 = v_4$ in $O(P^2)$ ways, and then by (2.1) we have

$$(x + h)^3 + u_3^3 = x^3 + u_1^3 \quad \text{and} \quad (y + g)^3 + v_3^3 = y^3 + v_1^3.$$

Since solutions with $x = u_3$ and $y = v_3$ are counted by $T(P)$, we may apply Lemma 2.2, together with Hua's Lemma, to deduce that there are $O(P^{10/3+\varepsilon})$ choices for $x, y, h, g, u_1, u_3, v_1$, and v_3. The case where $C = 0$ and $D \neq 0$ is identical.

It remains to consider solutions for which both C and D (and hence A and B) are non-zero. We first observe that, after substituting from (2.5) and completing the square, the first and fourth equations in (2.1) become

$$h(3x(x + h) + h^2) - \tfrac{1}{4}(A^3 + B^3) = 3A(u_3 + \tfrac{1}{2}A)^2 + 3B(u_4 + \tfrac{1}{2}B)^2 \qquad (2.6)$$

and

$$g(3y(y + g) + g^2) - \tfrac{1}{4}(C^3 + D^3) = 3C(v_3 + \tfrac{1}{2}C)^2 + 3D(v_4 + \tfrac{1}{2}D)^2, \qquad (2.7)$$

respectively. In view of (2.6) and (2.7), we further classify solutions according to whether

$$h(3x(x + h) + h^2) - \tfrac{1}{4}(A^3 + B^3) = 0 \qquad (2.8)$$

or

$$g(3y(y + g) + g^2) - \tfrac{1}{4}(C^3 + D^3) = 0. \qquad (2.9)$$

If both (2.8) and (2.9) hold, then we start by selecting values for A and B from among $O(P^2)$ possibilities, and (2.8) then determines h and x to $O(P^\varepsilon)$. Trivially, there are $O(P)$ choices for g, and (2.4) then determines C and D to $O(P^\varepsilon)$, whence y is determined to $O(1)$ by (2.9). Finally, u_3 and v_3 may be assigned in $O(P^2)$ ways, and this choice determines $\mathbf{u}$ and $\mathbf{v}$ to $O(1)$ in light of (2.5), (2.6), and (2.7). Hence there are $O(P^{5+\varepsilon})$ solutions of this type.

If (2.8) holds but (2.9) does not, then we assign A, B, and u_3 in $O(P^3)$ ways, so that $\mathbf{u}$ is determined to $O(1)$ by (2.6). Then h and x are again determined up to $O(P^\varepsilon)$, and there are $O(P^2)$ choices for y and g. This latter choice determines C and D to $O(P^\varepsilon)$ by (2.4), and we may apply Lemma 2.3 to (2.7), regarded as a binary quadratic equation in the variables v_3 and v_4. The case where (2.9) holds but (2.8) does not is exactly symmetric, so we see that there are $O(P^{5+\varepsilon})$ solutions of these two types.

Finally, if neither (2.8) nor (2.9) holds, then we fix h, C, and D in $O(P^3)$ ways, from which g, A, and B are determined to $O(P^\varepsilon)$ by (2.4). There are $O(P^2)$ possibilities for x and y, and Lemma 2.3 then shows that $\mathbf{u}$ and $\mathbf{v}$ are determined up to $O(P^\varepsilon)$ by (2.6) and (2.7). Thus we conclude that $N_4 \ll P^{16/3+\varepsilon}$.

(v) Now let γ be a parameter at our disposal, write $M = P^{1+\beta}$, and let N_5 be the number of solutions counted by $N(P)$ in which

$$hg \neq 0, \quad |hg/d| > M, \quad X_1 + X_2 \neq X_3 + X_4, \tag{2.10}$$

and $d = (h, g) \leq P^\gamma$. By symmetry, we may assume that $|h| \geq |g|$, the argument in the other case being identical. Write $h' = h/d$ and $g' = g/d$, so that $(h', g') = 1$. For any given d and $|h'| \geq |g'|$, we divide both sides of (2.3) by d^3 and apply Lemma 2.2 to deduce that there are then $O((|h'|P)^{4/3+\varepsilon})$ possible choices for $X_1, \ldots, X_4$. With X_i now fixed and $(h', g') = 1$, any two choices for u_i must be congruent modulo h', so one has $O(P/|h'|)$ possibilities for each of $u_1, \ldots, u_4$, and this determines $\mathbf{v}$. Since x and y are then determined by polynomials, we find that

$$
\begin{aligned}
N_5 &\ll \sum_{d \leq P^\gamma} \sum_{1 \leq g \leq P/d} \sum_{h \geq \max(g, M/gd)} (hP)^{4/3+\varepsilon}(P/h)^4 \\
&\ll P^{16/3+\varepsilon} \sum_{d \leq P^\gamma} \left(\sum_{g \leq (M/d)^{1/2}} \sum_{h \geq M/gd} h^{-8/3} + \sum_{g > (M/d)^{1/2}} \sum_{h \geq g} h^{-8/3} \right) \\
&\ll P^{16/3+\varepsilon} \sum_{d \leq P^\gamma} \left(\sum_{g \leq (M/d)^{1/2}} (M/gd)^{-5/3} + \sum_{g > (M/d)^{1/2}} g^{-5/3} \right) \\
&\ll P^{16/3+\varepsilon} \sum_{d \leq P^\gamma} \left((M/d)^{-5/3}(M/d)^{4/3} + (M/d)^{-1/3} \right) \\
&\ll P^{16/3+\varepsilon} M^{-1/3} \sum_{d \leq P^\gamma} d^{1/3} \ll P^{\frac{16}{3}-\frac{1}{3}(1+\beta)+\frac{4}{3}\gamma+\varepsilon}.
\end{aligned}
$$

(vi) Finally, let N_6 be the number of solutions counted by $N(P)$ with (2.10) and $d > P^\gamma$. In this case we use an affine slicing approach almost exactly as in Wooley [18]. As before, we exploit the symmetry of our system to focus attention on solutions with $|h| \geq |g|$. On recalling (2.3), we have that

$$X_1^3 + X_2^3 = X_3^3 + X_4^3 \quad \text{and} \quad X_1 + X_2 = X_3 + X_4 + H \tag{2.11}$$

for some integer H. For convenience, we write $X_i' = X_i/d$ and $H' = H/d$. For fixed h, g, and $\mathbf{u}$, one has

$$H' = g'(u_1 + u_2 - u_3 - u_4) - h'(v_1 + v_2 - v_3 - v_4),$$

which determines the residue class of H' modulo h'. Furthermore, since $|h'| \geq |g'|$, one has $|H'| \leq 4|h'|P$. Now from the equations (2.11), we find that

$$(X_1 + X_2 - X_3)^3 - (X_1^3 + X_2^3 - X_3^3) = (X_4 + H)^3 - X_4^3,$$

which simplifies to

$$3(X_1 - X_3)(X_2 - X_3)(X_1 + X_2) = H(3X_4^2 + 3X_4 H + H^2). \tag{2.12}$$

By (2.10), we have $H \neq 0$, so after dividing both sides of (2.12) by d^3 we see that at least one of $X_1' - X_3'$, $X_2' - X_3'$, or $X_1' + X_2'$ has a divisor $e \gg |H'|^{1/3}$ in common with H'. We suppose that

$$e = (H', X_1' - X_3') \gg |H'|^{1/3}$$

and write $X_1 - X_3 = deY$, the analysis in the other two cases being identical. Then, for fixed d and e, there are $O(|h'|P/e)$ choices for Y. Now, on substituting $X_4 = X_1 + X_2 - X_3 - H$ and $X_3 = X_1 - deY$ in (2.12), we obtain

$$3deY X_1^2 - 3(deY - H)X_2^2 - 3(deY)^2 X_1 - 3(deY - H)^2 X_2 = (deY - H)^3 - (deY)^3,$$

and after completing the square this becomes

$$3deY(X_1 - \tfrac{1}{2}deY)^2 - 3(deY - H)(X_2 + \tfrac{1}{2}(deY - H))^2 = \tfrac{1}{4}(deY - H)^3 - \tfrac{1}{4}(deY)^3.$$

Since $H \neq 0$, the quantities deY, $deY - H$, and $(deY - H)^3 - (deY)^3$ are all non-zero, so Lemma 2.3 may be applied. Thus, for fixed d, e, H, and Y, the values of X_1 and X_2 are determined up to $O(P^\varepsilon)$, and this fixes X_3 and X_4. For fixed g, h, and $\mathbf{u}$, this determines $\mathbf{v}$, and y is then determined to $O(1)$ by a polynomial. Thus we have

$$N_6 \ll \sum_{d > P^\gamma} \sum_{1 \leq h', g' \leq P/d} \sum_{x, \mathbf{u}} \sum_{H', e} \frac{h' P^{1+\varepsilon}}{e}.$$

For fixed H', a divisor estimate shows that there are $O(P^\varepsilon)$ possible choices for e, and for fixed h and x Lemma 2.1 shows that there are $O(P^{11/6+\varepsilon})$ choices for $\mathbf{u}$. Thus on summing trivially over g' and x we find that

$$N_6 \ll P^{1+\varepsilon} \sum_{d > P^\gamma} \frac{P}{d} \sum_{1 \leq h' \leq P/d} h' P \cdot P^{11/6+\varepsilon} \sum_{1 \leq H' \leq 4h'P} (H')^{-1/3},$$

on recalling that $e \gg |H'|^{1/3}$. We now divide the sum over $H' > h'$ into dyadic intervals and consider the sum over $H' \leq h'$ separately. Since the choice of h, g, and $\mathbf{u}$ fixes the residue class of H' modulo h', there are $O(1 + P/2^r)$ choices for H' satisfying $4h'P/2^{r+1} < H' \leq 4h'P/2^r$. Thus we have

$$\sum_{1 \leq H' \leq 4h'P} (H')^{-1/3} \ll (h')^{2/3} + \sum_{r=0}^{\infty} \frac{P}{2^r} \left(\frac{h'P}{2^{r+1}} \right)^{-1/3} \ll (h')^{2/3} + P^{2/3}(h')^{-1/3},$$

and here the second term is dominant whenever $h' \leq P^{2/3}$, which certainly holds when $\gamma \geq 1/3$. Subject to this condition, we finally obtain

$$N_6 \ll P^{11/2+\varepsilon} \sum_{d > P^\gamma} d^{-1} \sum_{h' \leq P/d} (h')^{2/3}$$

$$\ll P^{43/6+\varepsilon} \sum_{d > P^\gamma} d^{-8/3} \ll P^{43/6 - \frac{5}{3}\gamma + \varepsilon}.$$

We now choose an optimal value of $\gamma \geq 1/3$. In order that N_5 and N_6 have the same order of magnitude, we set

$$\frac{16}{3} - \frac{1}{3}(1 + \beta) + \frac{4}{3}\gamma = \frac{43}{6} - \frac{5}{3}\gamma,$$

which yields $\gamma = \frac{11}{18} + \frac{1}{9}(1 + \beta)$. In view of our bound for N_3, we choose β by setting

$$\frac{35}{6} + \beta = \frac{43}{6} - \frac{5}{3}\left(\frac{11}{18} + \frac{1}{9}(1 + \beta) \right),$$

which gives $\beta = \frac{7}{64}$ and $\gamma = \frac{47}{64}$. The result of Theorem 2 now follows immediately on assembling the bounds for $N_1, \ldots, N_6$ and noting that $\frac{1}{6} - \frac{7}{64} = \frac{11}{192}$.

3. Further Mean Value Estimates

Here we use the result of the previous section to obtain estimates for higher moments, which will be required in our application of the Hardy-Littlewood method in Section 5. As usual, the sharpest estimates are obtained by considering solutions in which some of the variables have no large prime factors. Thus when P and R are positive integers, write

$$\mathcal{A}(P,R) = \{n \in [1,P] \cap \mathbb{Z} : p|n, \ p \text{ prime} \Rightarrow p \leq R\}$$

for the set of R-smooth numbers up to P, and define the exponential sum

$$f(\boldsymbol{\alpha}; P, R) = \sum_{x,y \in \mathcal{A}(P,R)} e(\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 xy^2 + \alpha_3 y^3).$$

It will also be useful to have some variables in a complete interval, so we define

$$F(\boldsymbol{\alpha}; P) = \sum_{1 \leq x,y \leq P} e(\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 xy^2 + \alpha_3 y^3).$$

When there is no danger of confusion, we shall write $f(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha}; P, R)$ and $F(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}; P)$. Further, let

$$S_{s,r}(P, R) = \int_{\mathbb{T}^4} |F(\boldsymbol{\alpha})|^{2r} |f(\boldsymbol{\alpha})|^{2s} \, d\boldsymbol{\alpha}.$$

We adopt the convention that any statement involving ε and R means that for each $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that the assertion holds whenever $R \leq P^\eta$. In this section, our implicit constants will depend at most on ε unless otherwise noted. We start with an estimate for $S_{3,2}(P, R)$.

Lemma 3.1. *One has*

$$S_{3,2}(P, R) \ll P^{12 + \frac{1}{20} + \varepsilon}.$$

Proof. Define the difference operator Δ_1^* by

$$\Delta_1^*(f(x,y); h, g) = f(x+h, y+g) - f(x,y).$$

Then by Cauchy's inequality, one has

$$\begin{aligned}
S_{3,2}(P, R) &= \int_{\mathbb{T}^4} \left| \sum_{x,h} \sum_{y,g} e\left(\sum_{i=0}^3 \alpha_i \Delta_1^*(x^{3-i} y^i; h, g) \right) \right|^2 |f(\boldsymbol{\alpha})|^6 \, d\boldsymbol{\alpha} \\
&\ll P^2 \sum_{h,g} \int_{\mathbb{T}^4} \left| \sum_{x,y} e\left(\sum_{i=0}^3 \alpha_i \Delta_1^*(x^{3-i} y^i; h, g) \right) \right|^2 |f(\boldsymbol{\alpha})|^6 \, d\boldsymbol{\alpha},
\end{aligned}$$

and hence

$$S_{3,2}(P, R) \ll P^2 U_{3,2}(P, R), \tag{3.1}$$

where $U_{s,2}(P, R)$ denotes the number of solutions of the system

$$3h(x_1^2 - x_2^2 + h(x_1 - x_2)) = \sum_{i=1}^{s}(u_i^3 - u_{s+i}^3)$$

$$h(2(x_1y_1 - x_2y_2) + h(y_1 - y_2)) + g(x_1^2 - x_2^2 + 2h(x_1 - x_2)) = \sum_{i=1}^{s}(u_i^2 v_i - u_{s+i}^2 v_{s+i})$$

$$g(2(x_1y_1 - x_2y_2) + g(x_1 - x_2)) + h(y_1^2 - y_2^2 + 2g(y_1 - y_2)) = \sum_{i=1}^{s}(u_i v_i^2 - u_{s+i}v_{s+i}^2)$$

$$3g(y_1^2 - y_2^2 + g(y_1 - y_2)) = \sum_{i=1}^{s}(v_i^3 - v_{s+i}^3)$$

with

$$1 \le x_i, y_i \le P, \quad u_i, v_i \in \mathcal{A}(P, R), \quad \text{and} \quad |h|, |g| < P. \tag{3.2}$$

The argument is now very similar to the proof of Theorem 2 given in the previous section.

(i) Let U_1 denote the number of solutions counted by $U_{3,2}(P, R)$ for which

$$h(x_1 - x_2)(x_1 + x_2 + h) = g(y_1 - y_2)(y_1 + y_2 + g) = 0. \tag{3.3}$$

In this case, there are $O(P^4)$ choices for h, g, $\mathbf{x}$, and $\mathbf{y}$, and one sees that the number of choices for $\mathbf{u}$ and $\mathbf{v}$ is then

$$\int_{\mathbb{T}^4} |f(\boldsymbol{\alpha})|^6 \, d\boldsymbol{\alpha} \ll P^6,$$

on recalling Theorem 2 and considering the underlying Diophantine equations. Thus we have $U_1 \ll P^{10}$.

(ii) Let U_2 denote the number of solutions counted by $U_{3,2}(P, R)$ for which exactly one of h or g is zero and (3.3) does not hold. First suppose that $h = 0$ and $g \neq 0$. Then by Vaughan [13], Theorem 4.4, one has $O(P^{13/4+\varepsilon})$ choices for $\mathbf{u}$, and by a trivial estimate there are $O(P^3)$ choices for g and $\mathbf{y}$. Now for fixed g and $\mathbf{y}$, [13] again shows that there are $O(P^{13/4+\varepsilon})$ choices for $\mathbf{v}$. Finally, since $g(y_1 - y_2) \neq 0$, x_1 and x_2 are determined to $O(P^\varepsilon)$ by the second and third equations above. The situation when $g = 0$ and $h \neq 0$ is identical. Thus we see that $U_2 \ll P^{19/2+\varepsilon}$.

(iii) Let U_3 denote the number of solutions counted by $U_{3,2}(P, R)$ for which $hg \neq 0$,

$$(x_1 - x_2)(y_1 - y_2)(x_1 + x_2 + h)(y_1 + y_2 + g) = 0,$$

and (3.3) does not hold. If $x_1 = x_2$, then there are $O(P^3)$ choices for h, g, and $\mathbf{x}$ and $O(P^{13/4+\varepsilon})$ choices for $\mathbf{u}$. Now, as in the previous section, we find that

$$(gu_1 - hv_1)^3 + (gu_2 - hv_2)^3 + (gu_3 - hv_3)^3$$
$$= (gu_4 - hv_4)^3 + (gu_5 - hv_5)^3 + (gu_6 - hv_6)^3 \tag{3.4}$$

so by Hua [9], Theorem 4, there are $O(P^{7/2+\varepsilon})$ choices for $\mathbf{v}$, and then $\mathbf{y}$ is determined to $O(P^\varepsilon)$ by a divisor estimate. The other cases are handled similarly, and thus $U_3 \ll P^{39/4+\varepsilon}$.

(iv) Write $d = (h, g)$, let β be a parameter at our disposal, and let U_4 denote the number of solutions counted by $U_{3,2}(P, R)$ for which

$$0 < |hg/d| \leq P^{5/4+\beta} \quad \text{and} \quad (x_1 - x_2)(y_1 - y_2)(x_1 + x_2 + h)(y_1 + y_2 + g) \neq 0.$$

In this case, there are $O(P^{5/4+\beta})$ choices for the integer hg/d, whence by a divisor estimate there are $O(P^{5/4+\beta+\varepsilon})$ choices for h and g. Trivially, there are $O(P^2)$ choices for $\mathbf{x}$, and then we have $O(P^{13/4+\varepsilon})$ choices for $\mathbf{u}$. Thus by applying Hua [9] to (3.4), we see that there are $O(P^{7/2+\varepsilon})$ choices for $\mathbf{v}$, and then $\mathbf{y}$ is determined to $O(P^\varepsilon)$. Hence $U_4 \ll P^{10+\beta+\varepsilon}$.

(v) Now let γ be a parameter at our disposal, write $M = P^{5/4+\beta}$, and let U_5 be the number of solutions counted by $U_{3,2}(P, R)$ in which

$$|hg/d| > M, \quad (x_1 - x_2)(y_1 - y_2)(x_1 + x_2 + h)(y_1 + y_2 + g) \neq 0, \tag{3.5}$$

and $d = (h, g) \leq P^\gamma$. As before, we assume that $|h| \geq |g|$ and write $h' = h/d$ and $g' = g/d$, so that $(h', g') = 1$. For any given d and $|h'| \geq |g'|$, we divide both sides of (3.4) by d^3; then by Hua's Lemma there are $O((|h'|P)^{7/2+\varepsilon})$ possible choices for $X_1, \dots, X_6$, where $X_i = gu_i - hv_i$. With X_i now fixed and $(h', g') = 1$, any two choices for u_i must be congruent modulo h', so one has $O(P/|h'|)$ possibilities for each of $u_1, \dots, u_6$, and this determines $\mathbf{v}$. Since $\mathbf{x}$ and $\mathbf{y}$ are then determined to $O(P^\varepsilon)$, we find that

$$
\begin{aligned}
U_5 &\ll \sum_{d \leq P^\gamma} \sum_{1 \leq g \leq P/d} \sum_{h \geq \max(g, M/gd)} (hP)^{7/2+\varepsilon}(P/h)^6 \\
&\ll P^{19/2+\varepsilon} \sum_{d \leq P^\gamma} \left(\sum_{g \leq (M/d)^{1/2}} \sum_{h \geq M/gd} h^{-5/2} + \sum_{g > (M/d)^{1/2}} \sum_{h \geq g} h^{-5/2} \right) \\
&\ll P^{19/2+\varepsilon} \sum_{d \leq P^\gamma} \left(\sum_{g \leq (M/d)^{1/2}} (M/gd)^{-3/2} + \sum_{g > (M/d)^{1/2}} g^{-3/2} \right) \\
&\ll P^{19/2+\varepsilon} \sum_{d \leq P^\gamma} \left((M/d)^{-3/2}(M/d)^{5/4} + (M/d)^{-1/4} \right) \\
&\ll P^{19/2+\varepsilon} M^{-1/4} \sum_{d \leq P^\gamma} d^{1/4} \ll P^{\frac{19}{2} - \frac{1}{4}(\frac{5}{4}+\beta) + \frac{5}{4}\gamma + \varepsilon}.
\end{aligned}
$$

(vi) Finally, let U_6 be the number of solutions counted by $U_{3,2}(P, R)$ with (3.5) and $d > P^\gamma$. For fixed d, there are at most $(P/d)^2$ choices for h and g and P^2 choices for $\mathbf{x}$. Now, for given h and $\mathbf{x}$, Theorem 4.4 of Vaughan [13] shows that there are $O(P^{13/4+\varepsilon})$ choices for $\mathbf{u}$, and then on recalling (3.4) and Hua [9], Theorem 4, we see that there are $O(P^{7/2+\varepsilon})$ choices for $\mathbf{v}$, from which $\mathbf{y}$ is determined to $O(P^\varepsilon)$. Thus we have

$$U_6 \ll P^{43/4+\varepsilon} \sum_{d > P^\gamma} d^{-2} \ll P^{43/4-\gamma+\varepsilon}.$$

To optimize the results of (v) and (vi), we set

$$9 + \frac{3}{16} - \frac{1}{4}\beta + \frac{5}{4}\gamma = 10 + \frac{3}{4} - \gamma,$$

which gives $\gamma = \frac{25}{36} + \frac{1}{9}\beta$. Now on recalling (iv), we choose β so that

$$10 + \beta = \frac{43}{4} - \frac{25}{36} - \frac{1}{9}\beta,$$

which gives $\beta = \frac{1}{20}$. Hence $U_{3,2}(P,R) \ll P^{10+\frac{1}{20}}$, and the lemma follows on recalling (3.1). $\square$

Before proceeding, we record an easy consequence of Lemma 3.1.

Lemma 3.2. *One has*

$$S_{2,2}(P,R) \ll P^{9+\frac{1}{40}+\varepsilon}.$$

Proof. By Theorem 2, we have

$$\int_{\mathbb{T}^4} |F(\boldsymbol{\alpha})|^6 d\boldsymbol{\alpha} \ll P^6.$$

Thus by applying the Cauchy-Schwarz inequality and Lemma 3.1 we obtain

$$S_{2,2}(P,R) \ll \left(\int_{\mathbb{T}^4} |F(\boldsymbol{\alpha})|^6 d\boldsymbol{\alpha}\right)^{1/2} \left(\int_{\mathbb{T}^4} |F(\boldsymbol{\alpha})|^2 |f(\boldsymbol{\alpha})|^8 d\boldsymbol{\alpha}\right)^{1/2} \ll P^{9+\frac{1}{40}+\varepsilon}$$

on considering the underlying Diophantine equations. $\square$

We now proceed to estimate some higher moments.

Lemma 3.3. *One has*

$$S_{4,2}(P,R) \ll P^{15+\frac{1}{3}+\varepsilon}.$$

Proof. By Cauchy's inequality, we have $S_{4,2}(P,R) \ll P^2 U_{4,2}(P,R)$, and the estimation of $U_{4,2}(P,R)$ proceeds almost exactly as in the proof of Lemma 3.1. The only modifications are that we use Lemma 3.2 in place of Theorem 2 in the analysis of case (i) and we replace the 6th moment estimates of $P^{13/4+\varepsilon}$ and $P^{7/2+\varepsilon}$ by Hua's 8th moment estimate of $P^{5+\varepsilon}$. Taking $M = P^{4/3}$ and $\gamma = 2/3$ produces identical bounds for the final three cases and hence gives an optimal result. $\square$

We remark that the argument of the preceding proof in fact shows that one may replace $S_{4,2}(P,R)$ by $S_{2,4}(P,R)$ in the statement of Lemma 3.3. Further, we note that tiny improvements in the exponents of Lemmata 3.1 and 3.2 may be achieved by using the results of Wooley [17], [20] in place of Vaughan [13], but such improvements do not have significant consequences in the present application.

For higher moments, we apply the (single) efficient differencing procedure of [10]. Although our methods always allow us to take a few variables ranging over a complete interval, we will often simplify by stating results for mean values in which all the variables are smooth. Thus we adopt the notation of writing $S_s(P,R)$ for $S_{s,0}(P,R)$. Further, we say that Δ_s is an admissible exponent if one has $S_s(P,R) \ll P^{\lambda_s+\varepsilon}$, where $\lambda_s = 4s - 12 + \Delta_s$, and in this situation we call λ_s a permissible exponent. To this point we have obtained the admissible exponents

$$\Delta_3 = 6, \quad \Delta_4 = 5\tfrac{1}{40}, \quad \Delta_5 = 4\tfrac{1}{20}, \quad \text{and} \quad \Delta_6 = 3\tfrac{1}{3}. \tag{3.6}$$

The above method of generating admissible exponents becomes noticeably less effective when $s > 6$, since the maximum savings of P^3 in estimating the number of solutions of

$$u_1^3 + \cdots + u_t^3 = u_{t+1}^3 + \cdots + u_{2t}^3$$

is already achieved when $t = 4$. Since the results of [10] are directly applicable only for $s \geq 11$, we will need the following lemma to work out admissible exponents when s lies in the intermediate range.

Lemma 3.4. *One has*

$$S_{s+2}(P, R) \ll P^5 S_s(P, R) + P^{19/6+\varepsilon} S_{s-1,2}(P, R) + P^{\frac{2}{3}s+6+\varepsilon} S_s(P^{5/6}, R).$$

Proof. This follows on using [10], Lemma 3.2, in the initial stages of the argument of [10], Lemma 5.1. □

We now apply Lemma 3.4 repeatedly to obtain admissible exponents Δ_s for $7 \leq s \leq 12$. First of all, by using Lemma 3.3 and making a trivial estimate, we see that

$$S_{5,2}(P, R) \ll P^{19+\frac{1}{3}+\varepsilon},$$

and using this together with Lemma 3.3 in Lemma 3.4 gives

$$S_8(P, R) \ll P^{22+\frac{7}{9}+\varepsilon}. \tag{3.7}$$

Now using the Cauchy-Schwarz inequality to interpolate between $S_{4,2}$ and S_8, we obtain

$$S_7(P, R) \ll (S_{4,2}(P, R))^{1/2}(S_8(P, R))^{1/2} \ll P^{19+\frac{1}{18}+\varepsilon}. \tag{3.8}$$

Putting (3.7) and (3.8) into Lemma 3.4 now yields

$$S_9(P, R) \ll P^{26+\frac{59}{108}+\varepsilon} \quad \text{and} \quad S_{10}(P, R) \ll P^{30+\frac{17}{54}+\varepsilon},$$

on recalling Lemma 3.3 and noting the trivial inequality

$$S_{t,2}(P, R) \leq P^8 S_t(P, R).$$

Continuing the iteration, we find that

$$S_{11}(P, R) \ll P^{34+\frac{79}{648}+\varepsilon} \quad \text{and} \quad S_{12}(P, R) \ll P^{37+\frac{301}{324}+\varepsilon},$$

and thus we have the admissible exponents

$$\begin{gathered} \Delta_7 = 3\tfrac{1}{18}, \quad \Delta_8 = 2\tfrac{7}{9}, \quad \Delta_9 = 2\tfrac{59}{108}, \\ \Delta_{10} = 2\tfrac{17}{54}, \quad \Delta_{11} = 2\tfrac{79}{648}, \quad \Delta_{12} = \tfrac{625}{324}. \end{gathered} \tag{3.9}$$

Further admissible exponents can now be read off from Lemma 5.1 of [10]. Namely, if $s \geq 11$ and Δ_s is admissible then the exponent

$$\Delta_{s+2t} = \Delta_s(5/6)^t \tag{3.10}$$

is also admissible.

4. Weyl Differencing

Here we obtain estimates for the modulus of the exponential sum $F(\boldsymbol{\alpha})$ when at least one of the α_j is badly approximated by rationals. In [10], estimates of this type were obtained for $f(\boldsymbol{\alpha})$ by using the large sieve to relate the modulus of the sum to known mean values. This treatment allowed us to obtain bounds of the form $P^{2-\sigma(k)+\varepsilon}$, where $\sigma(k)^{-1} \asymp k^3 \log k$, and for large k this is substantially better than the exponential decay that results from Weyl differencing. For $k = 3$, however, we are much better off applying a two-fold Weyl differencing procedure. For purposes of application, it is useful to consider the slightly more general exponential sum

$$F(\boldsymbol{\alpha}; P, Q) = \sum_{1 \le x \le P} \sum_{1 \le y \le Q} e(\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 x y^2 + \alpha_3 y^3).$$

Lemma 4.1. *Suppose that $Q \asymp P$ and that for some j we have $q_j \in \mathbb{N}$ and $a_j \in \mathbb{Z}$ with*

$$(q_j, a_j) = 1 \quad and \quad |q_j \alpha_j - a_j| \le q_j^{-1}. \tag{4.1}$$

Then one has

$$|F(\boldsymbol{\alpha}; P, Q)| \ll P^{2+\varepsilon}(q_j^{-1} + P^{-1} + q_j P^{-3})^{1/4}.$$

Proof. First suppose that (4.1) holds with $j = 0$. Then by Weyl's inequality (see for instance Lemma 2.4 of Vaughan [14]) one has

$$\begin{aligned}
|F(\boldsymbol{\alpha}; P, Q)| &\le \sum_{y \le Q} \left| \sum_{x \le P} e(\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 x y^2) \right| \\
&\ll QP^{1+\varepsilon}(q_0^{-1} + P^{-1} + q_0 P^{-3})^{1/4},
\end{aligned}$$

and the result follows. Note that if instead (4.1) holds with $j = 3$, then we obtain the same conclusion simply by interchanging the roles of x and y in the above argument.

Now suppose that (4.1) holds with $j = 1$. Then by Cauchy's inequality we have

$$\begin{aligned}
|F(\boldsymbol{\alpha})|^2 &\le Q \sum_{y \le Q} \left| \sum_{x \le P} e(\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 x y^2) \right|^2 \\
&\ll P \sum_{y \le Q} \left(P + \sum_{x,h} e(\alpha_0(3x^2 h + 3xh^2 + h^3) + \alpha_1 y(2xh + h^2) + \alpha_2 y^2 h) \right) \\
&\ll P^3 + P \sum_{y \le Q} \sum_{x,h} e(\alpha_0(3x^2 h + 3xh^2 + h^3) + \alpha_1 y(2xh + h^2) + \alpha_2 y^2 h),
\end{aligned}$$

where the second sum is over x and $h \ne 0$ with $1 \le x \le P$ and $1 - x \le h \le P - x$, and where we have abbreviated $F(\boldsymbol{\alpha}; P, Q)$ by $F(\boldsymbol{\alpha})$. Then on using Cauchy's inequality again

we obtain

$$
\begin{aligned}
|F(\boldsymbol{\alpha})|^4 \;\ll\;& P^6 + P^4 \sum_{x,h} \left| \sum_{y \le Q} e(\alpha_0(3x^2h + 3xh^2 + h^3) + \alpha_1 y(2xh + h^2) + \alpha_2 y^2 h) \right|^2 \\
\ll\;& P^6 + P^4 \sum_{x,h} \left(P + \sum_{y,g} e(\alpha_1 g(2xh + h^2) + \alpha_2 h(2yg + g^2)) \right) \\
\ll\;& P^7 + P^4 \sum_{x,h,y,g} e(\alpha_1 g(2xh + h^2) + \alpha_2 h(2yg + g^2)) \\
\ll\;& P^7 + P^4 \sum_{1 \le |h|,|g| \le P} \sum_{y \in I(Q,g)} \left| \sum_{x \in I(P,h)} e(2\alpha_1 ghx) \right|,
\end{aligned}
$$

where $I(P,h) = [1, P] \cap [1 - h, P - h]$. Thus on summing the geometric progression, recalling a standard divisor function estimate, and using Lemma 2.2 of [14], we find that

$$
\begin{aligned}
|F(\boldsymbol{\alpha}; P, Q)|^4 \;\ll\;& P^7 + QP^4 \sum_{1 \le |h|,|g| \le P} \min(P, ||2\alpha_1 gh||^{-1}) \\
\ll\;& P^7 + QP^{4+\varepsilon} \sum_{n \le 2P^2} \min(P, ||\alpha_1 n||^{-1}) \\
\ll\;& P^7 + QP^{7+\varepsilon}(q_1^{-1} + P^{-1} + q_1 P^{-3}),
\end{aligned}
$$

whence

$$
|F(\boldsymbol{\alpha}; P, Q)| \ll P^{2+\varepsilon}(q_1^{-1} + P^{-1} + q_1 P^{-3})^{1/4}.
$$

Again, the same conclusion follows when (4.1) holds with $j = 2$ by repeating the argument with the roles of x and y reversed. $\qquad\square$

Next we record a consequence of the above lemma, which will be useful in our application of the circle method.

Lemma 4.2. *Let $\alpha_0, \ldots, \alpha_3$ be real numbers with the property that whenever there exist $q \in \mathbb{N}$ and $a_0, \ldots, a_3 \in \mathbb{Z}$ with $(q, a_0, \ldots, a_3) = 1$ and $|q\alpha_j - a_j| \le P^{\delta-3}$ one has $q > P^\delta$. Then whenever $Q \asymp P$ one has*

$$
|F(\boldsymbol{\alpha}; P, Q)| \ll P^{2 - \delta/16 + \varepsilon}.
$$

Proof. Let $\alpha_0, \ldots, \alpha_3$ be as in the statement of the lemma, and write $\nu = \delta/4$. For each j, Dirichlet's Theorem allows us to find $q_j \in \mathbb{N}$ and $b_j \in \mathbb{Z}$ with $(q_j, b_j) = 1$ such that $|q_j \alpha_j - b_j| \le P^{\nu-3}$ and $q_j \le P^{3-\nu}$. If $q_j > P^\nu$ for some j, then the conclusion follows from Lemma 4.1. Otherwise, put $q = [q_0, \ldots, q_3]$ and $a_j = b_j q / q_j$. Then we have $(q, a_0, \ldots, a_3) = 1$ and $q \le q_j P^{3\nu} \le P^\delta$ for each j and hence

$$
|\alpha_j - a_j/q| \le q_j^{-1} P^{\nu-3} \le q^{-1} P^{\delta-3} \quad (0 \le j \le 3),
$$

contradicting the hypothesis of the lemma. $\qquad\square$

5. The Circle Method

Now we are in a position to prove Theorem 1 by applying the circle method along the lines of [10], §10. The following lemma provides us with non-singular local solutions to (1.1).

Lemma 5.1. *If $s \geq 14$ and $c_1, \ldots, c_s$ are non-zero integers, then the system (1.1) has a non-singular real solution and a non-singular p-adic solution for all primes p.*

Proof. After setting

$$y_1 = \cdots = y_7 = 0, \quad x_8 = \cdots = x_{14} = 0, \quad \text{and} \quad x_i = y_i = 0 \quad (i > 14),$$

the system (1.1) reduces to

$$c_1 x_1^3 + \cdots + c_7 x_7^3 = 0 \quad \text{and} \quad c_8 y_8^3 + \cdots + c_{14} y_{14}^3 = 0.$$

By a well-known result of Baker [2], each of these equations has a non-trivial integral solution; suppose that $\mathbf{x}$ and $\mathbf{y}$ are solutions with x_I and y_J non-zero. Then on writing

$$\psi_j(\mathbf{x}, \mathbf{y}) = c_1 x_1^{3-j} y_1^j + \cdots + c_s x_s^{3-j} y_s^j \quad (0 \leq j \leq 3),$$

we have

$$\det \left(\frac{\partial \psi_j}{\partial x_I}, \frac{\partial \psi_j}{\partial y_I}, \frac{\partial \psi_j}{\partial x_J}, \frac{\partial \psi_j}{\partial y_J} \right)_{0 \leq j \leq 3} = (3c_I c_J)^2 x_I^4 y_J^4 \neq 0.$$

Thus $(\mathbf{x}, \mathbf{y})$ is a non-singular integer solution of (1.1), so it is non-singular in each local field as well. $\qquad\square$

By Lemma 5.1 and [10], Lemma 10.2, we may assume that the system (1.1) has a non-singular real solution $(\boldsymbol{\eta}, \boldsymbol{\xi})$ with $0 < |\eta_i|, |\xi_i| < \frac{1}{2}$ for $i = 1, \ldots, s$. For each i, we write

$$\eta_i^+ = \eta_i + \tfrac{1}{2}|\eta_i| \quad \text{and} \quad \eta_i^- = \eta_i - \tfrac{1}{2}|\eta_i|$$

and

$$\xi_i^+ = \xi_i + \tfrac{1}{2}|\xi_i| \quad \text{and} \quad \xi_i^- = \xi_i - \tfrac{1}{2}|\xi_i|.$$

Now let P be a large positive number, put $R = P^\eta$ with $\eta \leq \eta_0(\varepsilon)$, and let $c_1, \ldots, c_s$ be non-zero integers. Throughout this section, the implicit constants arising in our analysis may depend on $c_1, \ldots, c_s$ and on the real solution $(\boldsymbol{\eta}, \boldsymbol{\xi})$. We define the exponential sums

$$F_i(\boldsymbol{\alpha}) = \sum_{\eta_i^- P < x \leq \eta_i^+ P} \sum_{\xi_i^- P < y \leq \xi_i^+ P} e(c_i(\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 x y^2 + \alpha_3 y^3))$$

and

$$f_i(\boldsymbol{\alpha}) = \sum_{\substack{\eta_i^- P < x \leq \eta_i^+ P \\ |x| \in \mathcal{A}(P,R)}} \sum_{\substack{\xi_i^- P < y \leq \xi_i^+ P \\ |y| \in \mathcal{A}(P,R)}} e(c_i(\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 x y^2 + \alpha_3 y^3)).$$

Further, write $s = t + 2u + v$ and define

$$\mathcal{F}(\boldsymbol{\alpha}) = \prod_{i=1}^{t} F_i(\boldsymbol{\alpha}) \quad \text{and} \quad \mathcal{G}(\boldsymbol{\alpha}) = \prod_{i=t+1}^{s} f_i(\boldsymbol{\alpha}).$$

Finally, let

$$R_s(P) = \int_{\mathbb{T}^4} \mathcal{F}(\boldsymbol{\alpha}) \mathcal{G}(\boldsymbol{\alpha}) \, d\boldsymbol{\alpha},$$

and observe that $R_s(P)$ is a lower bound for the number of integral solutions of (1.1) lying in the box $[-P, P]^{2s}$.

We dissect $\mathbb{T}^4$ into major and minor arcs as follows. Write $c = \max |c_i|$, let $\delta \in (0,1)$ be a parameter at our disposal, define

$$\mathfrak{M} = \bigcup_{\substack{1 \le a_0, \dots, a_3 \le q \le cP^\delta \\ (q, a_0, \dots, a_3) = 1}} \mathfrak{M}(q, \mathbf{a}),$$

where

$$\mathfrak{M}(q, \mathbf{a}) = \{\boldsymbol{\alpha} \in \mathbb{T}^4 : |q\alpha_i - a_i| \le P^{\delta-3} \ (0 \le i \le 3)\},$$

and put $\mathfrak{m} = \mathbb{T}^4 \setminus \mathfrak{M}$. It is easily seen that the $\mathfrak{M}(q, \mathbf{a})$ are disjoint whenever P is sufficiently large and $\delta < 1$.

We further define the pruned major arcs by

$$\mathfrak{N} = \bigcup_{\substack{1 \le a_0, \dots, a_3 \le q \le W \\ (q, a_0, \dots, a_3) = 1}} \mathfrak{N}(q, \mathbf{a}),$$

where $W < cP^\delta$ is a parameter at our disposal and

$$\mathfrak{N}(q, \mathbf{a}) = \{\boldsymbol{\alpha} \in \mathbb{T}^4 : |\alpha_i - a_i/q| \le WP^{-3} \ (0 \le i \le 3)\}.$$

The following pruning lemma is reminiscent of [16], Lemma 9.2.

Lemma 5.2. *If $t \ge 16$ and $\Delta_u < (1 - \delta)t$ is an admissible exponent, then one has*

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |F_i(\boldsymbol{\alpha})|^t |f_j(\boldsymbol{\alpha})|^{2u} d\boldsymbol{\alpha} \ll W^{-\sigma} P^{2t+4u-12}$$

for some $\sigma > 0$.

Proof. When $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{a})$, we have by a simple modification of [10], Lemma 8.1, that

$$F_i(\boldsymbol{\alpha}) = V_i(\boldsymbol{\alpha}) + O(P^{1+\delta}),$$

where

$$V_i(\boldsymbol{\alpha}) = q^{-2} S_i(q, \mathbf{a}) v_i(\boldsymbol{\alpha} - \mathbf{a}/q),$$

$$S_i(q, \mathbf{a}) = \sum_{1 \le x, y \le q} e\left(\frac{c_i(a_0 x^3 + a_1 x^2 y + a_2 xy^2 + a_3 y^3)}{q}\right),$$

and

$$v_i(\boldsymbol{\beta}) = \int_{\eta_i^- P}^{\eta_i^+ P} \int_{\xi_i^- P}^{\xi_i^+ P} e(c_i(\beta_0 \gamma^3 + \beta_1 \gamma^2 \nu + \beta_2 \gamma \nu^2 + \beta_3 \nu^3)) \, d\gamma \, d\nu.$$

Then whenever $\mathcal{M} \subset \mathfrak{M}$, we have

$$\int_{\mathcal{M}} |F_i(\boldsymbol{\alpha})|^t |f_j(\boldsymbol{\alpha})|^{2u} d\boldsymbol{\alpha} \ll P^{4u} \int_{\mathcal{M}} |V_i(\boldsymbol{\alpha})|^t d\boldsymbol{\alpha} + P^{t(1+\delta)} \int_{\mathbb{T}^4} |f_j(\boldsymbol{\alpha})|^{2u} d\boldsymbol{\alpha}.$$

On using the estimates

$$S_i(q, \mathbf{a}) \ll q^{2-1/3+\varepsilon}$$

and

$$v_i(\boldsymbol{\beta}) \ll P^2 \prod_{i=0}^{3} (1 + P^3 |\beta_i|)^{-1/12},$$

(see for example [1]), we find that whenever $t \geq 16$ one has

$$\int_{\mathfrak{M}\backslash\mathfrak{N}} |V_i(\boldsymbol{\alpha})|^t d\boldsymbol{\alpha} \ll P^{2t-12} W^{-1/3}.$$

Moreover, it is clear by considering the underlying Diophantine equations that all the mean value estimates from Section 3 hold with $f(\boldsymbol{\alpha})$ replaced by $f_j(\boldsymbol{\alpha})$, so we have

$$\int_{\mathbb{T}^4} |f_j(\boldsymbol{\alpha})|^{2u} d\boldsymbol{\alpha} \ll P^{4u-12+\Delta_u+\varepsilon} \ll P^{4u-12+t(1-\delta)-\gamma}$$

for some $\gamma > 0$, and the lemma follows. $\qquad\square$

We are now ready to derive the lower bound for $R_s(P)$. If $\boldsymbol{\alpha} \in \mathfrak{m}$, then by the argument of [10], Lemma 10.3, one sees that the hypotheses of Lemma 4.2 hold with $\boldsymbol{\alpha}$ replaced by $c_i\boldsymbol{\alpha}$ and hence

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |F_i(\boldsymbol{\alpha})| \ll P^{2-\delta/16+\varepsilon}.$$

Thus by Hölder's inequality we have

$$\begin{aligned}
\int_{\mathfrak{m}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})| \, d\boldsymbol{\alpha} &\ll P^{2t+2v-\frac{t\delta}{16}+\varepsilon} \prod_{i=t+1}^{t+2u} \left(\int_{\mathbb{T}^4} |f_i(\boldsymbol{\alpha})|^{2u} d\boldsymbol{\alpha}\right)^{1/2u} \\
&\ll P^{2s-12+\Delta_u-\frac{t\delta}{16}+\varepsilon} \ll P^{2s-12-\tau}
\end{aligned}$$

for some $\tau > 0$, provided that

$$t\delta > 16\Delta_u. \tag{5.1}$$

In view of the condition in Lemma 5.2, the optimal choice for δ satisfies $1 - \delta = \frac{\delta}{16}$, so we take $\delta = 16/17$. Now on recalling (3.9) and (3.10) we see that the exponent

$$\Delta_{16} = \frac{625}{324}\left(\frac{5}{6}\right)^2 < \frac{23}{17}$$

is admissible, and it transpires that choosing $t = 23$ and $u = 16$ minimizes the quantity $t + 2u$ subject to (5.1). Thus on taking $v \geq 2$, we have a total of $s \geq 57$ variables.

It now suffices to deal with the major arcs. By applying Lemma 5.2 and making a trivial estimate, we have for some i and j that

$$\int_{\mathfrak{M}\backslash\mathfrak{N}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})| \, d\boldsymbol{\alpha} \ll P^{2v} \int_{\mathfrak{M}\backslash\mathfrak{N}} |F_i(\boldsymbol{\alpha})|^{23} |f_j(\boldsymbol{\alpha})|^{32} d\boldsymbol{\alpha} \ll P^{2s-12} W^{-\sigma},$$

and so it suffices to deal with the pruned major arc $\mathfrak{N}$. But it follows immediately from the analysis of [10], §10 that

$$\int_{\mathfrak{N}} \mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha}) \, d\boldsymbol{\alpha} \gg P^{2s-12},$$

provided that W is taken to be a suitably small power of $\log P$.

Finally, we need to relate $R_s(P)$ to $L_s(P)$ by accounting for the possibility that distinct solutions of (1.1) correspond to different parameterizations of the same line. By the argument of [10], Lemma 10.12, we have

$$R_s(P, d) \ll \frac{P^{2v}}{d^v} \int_{\mathbb{T}^4} |\mathcal{F}(\boldsymbol{\alpha})| \left(\prod_{i=24}^{55} |f_i(\boldsymbol{\alpha})|\right) d\boldsymbol{\alpha} \ll \frac{P^{2s-12}}{d^2},$$

where $R_s(P, d)$ denotes the number of solutions counted by $R_s(P)$ for which $(x_1, \ldots, x_s) = d$. Thus on following through the corresponding argument in [10], we find that $L_s(P) \gg P^{2s-12}$, and this completes the proof of Theorem 1.

As mentioned in the introduction, essentially the same analysis may be applied to establish a result on the two-dimensional version of Waring's problem discussed in [10]. In that argument, we may clearly take $v = 0$, and it follows immediately that $G_1^*(3) \leq 55$.

References

[1] G. I. Arkhipov, A. A. Karatsuba, and V. N. Chubarikov, *Multiple trigonometric sums*, Trudy Mat. Inst. Steklov **151** (1980), 1–126.

[2] R. C. Baker, *Diagonal cubic equations II*, Acta Arith. **53** (1989), 217–250.

[3] B. J. Birch, *Homogeneous forms of odd degree in a large number of variables*, Mathematika **4** (1957), 102–105.

[4] R. Brauer, *A note on systems of homogeneous algebraic equations*, Bull. Amer. Math. Soc. **51** (1945), 749–755.

[5] T. Estermann, *Einige Sätze über quadratfreie Zahlen*, Math. Ann. **105** (1931), 653–662.

[6] D. R. Heath-Brown, *The density of rational points on cubic surfaces*, Acta Arith. **79** (1997), 17–30.

[7] C. Hooley, *On the representations of a number as the sum of four cubes*, Proc. London Math. Soc. **(3) 36** (1978), 117–140.

[8] ———, *On the numbers that are representable as the sum of two cubes*, J. reine angew. Math. **314** (1980), 146–173.

[9] L.-K. Hua, *Additive theory of prime numbers*, A.M.S., Providence, Rhode Island, 1965.

[10] S. T. Parsell, *Multiple exponential sums over smooth numbers*, J. reine angew. Math. (to appear).

[11] W. M. Schmidt, *On cubic polynomials III. Systems of p-adic equations*, Monatsh. Math. **93** (1982), 211–223.

[12] ———, *On cubic polynomials IV. Systems of rational equations*, Monatsh. Math. **93** (1982), 329–348.

[13] R. C. Vaughan, *A new iterative method in Waring's problem*, Acta Math. **162** (1989), 1–71.

[14] ———, *The Hardy-Littlewood method*, 2nd ed., Cambridge University Press, Cambridge, 1997.

[15] R. C. Vaughan and T. D. Wooley, *Further improvements in Waring's problem*, Acta Math. **174** (1995), 147–240.

[16] T. D. Wooley, *On simultaneous additive equations II*, J. reine angew. Math. **429** (1991), 141–198.

[17] ———, *Breaking classical convexity in Waring's problem: Sums of cubes and quasi-diagonal behaviour*, Invent. Math. **122** (1995), 421–451.

[18] ———, *Sums of two cubes*, Internat. Math. Res. Notices **4** (1995), 181–185.

[19] ———, *Linear spaces on cubic hypersurfaces and pairs of homogeneous cubic equations*, Bull. London Math. Soc. **29** (1997), 556–562.

[20] ———, *Sums of three cubes*, Mathematika (to appear).

Department of Mathematics, University of Michigan, East Hall, 525 East University Avenue, Ann Arbor, MI 48109-1109, U.S.A. (through August 15, 1999)

E-mail address: `stparsel@math.lsa.umich.edu`

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, U.S.A. (from August 15, 1999)

E-mail address: `parsell@alum.mit.edu`