# MAT 261—Final Exam A—5/8/14

Name: ____________________________________________

Calculators are not permitted. Show all of your work using correct mathematical notation.

1. (10 points) Find the area of the triangle with vertices $P(1, 1, 0)$, $Q(2, 3, 1)$, and $R(1, 4, 2)$.

2. (15 points) The position of a particle in two dimensions at time $t$ is given by the formula

$$\mathbf{c}(t) = (t^3 + 5)\mathbf{i} + (2t^3 - 7)\mathbf{j}.$$

(a) Determine the particle's speed at the instant when $t = 1$.

(b) Find the length of the particle's path over the interval $0 \leqslant t \leqslant 2$.

3. (10 points) Find parametric equations for the line that passes through the point $(1, 2, 3)$ and is perpendicular to the plane $4x + 5y + 6z = 7$.

4. (15 points) Consider the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.

   (a) Sketch the domain of $f$ and the level curve passing through the point $(1, -1)$.

   (b) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, -1)$.

5. (15 points) Find the directional derivative of

$$f(x, y, z) = ze^{xz} + \ln(x^2 + 4y)$$

at the point $(0, 1, 2)$ in the direction of the vector $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

6. (10 points) Determine the location of all local maxima, local minima, and saddle points of the function

$$f(x, y) = x^2 + y^2 - xy + x.$$

7. (10 points) Evaluate $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} e^{-(x^2+y^2)}\,dy\,dx$ by converting to polar coordinates. Include a sketch of the region of integration.

8. (15 points) Evaluate $\displaystyle\int_0^8 \int_{x^{1/3}}^2 \frac{1}{y^4+1}\,dy\,dx$ by reversing the order of integration. Include a sketch of the region of integration.

9. (10 points) Use a line integral to find the mass of a wire lying along the curve

$$\mathbf{c}(t) = (t^2 - 1)\mathbf{i} + 2t\mathbf{j} + 5\mathbf{k} \qquad (0 \leqslant t \leqslant 1),$$

if the density is $\delta(x, y, z) = 3y$ kg/m.

10. (15 points) A surface $\mathcal{S}$ has parametrization

$$G(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle \qquad (0 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant 2\pi).$$

(a) Which of the following best describes the surface? (Circle your answer.)

cone  cylinder  hemisphere  paraboloid  plane  sphere

(b) Find the outward-pointing normal vector $\mathbf{n}(r, \theta)$.

(c) Calculate the outward flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ for the vector field $\mathbf{F} = \langle xz, 0, 0\rangle$.

11. (13 points) Suppose that $\mathbf{F} = \langle x^3y^2, x - 2y\rangle$, and let $\mathcal{C}$ be the boundary of the region in the first quadrant enclosed by the curve $y = x^2$ and the lines $y = 0$ and $x = 1$, oriented counter-clockwise. Use Green's Theorem to find the circulation $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$.

12. (12 points) Use the Divergence Theorem to calculate the outward flux of the vector field $\mathbf{F} = \langle y^7, \cos(xz^5), z^3 + e^{xy}\rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

## The Fundamental Theorems of Vector Analysis

**Green's Theorem.** If $\mathcal{C}$ is a simple closed curve traversed counter-clockwise and $\mathcal{D}$ is the region enclosed by $\mathcal{C}$, then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \oint_{\mathcal{C}} F_1\,dx + F_2\,dy = \iint_{\mathcal{D}} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA,$$

provided that $F_1$ and $F_2$ are differentiable functions with continuous first partial derivatives.

**Stokes' Theorem.** If $\mathcal{S}$ is a smooth oriented surface, then

$$\oint_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\partial \mathcal{S}$ denotes the boundary of $\mathcal{S}$, oriented so that a normal vector "walking" along the curve has the surface on its left.

**The Divergence Theorem.** Let $\mathcal{S}$ be a closed surface, oriented with outward-pointing normal vectors, that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \mathrm{div}(\mathbf{F})\,dV,$$

provided that all points in $\mathcal{W}$ lie in the domain of $\mathbf{F}$.

We have

$$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$$

and

$$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F},$$

where

$$\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle.$$