MA 208—Final Exam—12/15/08

Name: Solutions Instructor: Parsell

Calculators are permitted, but you must show all of your work using correct notation.

1. (10 points) Find the angle between the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ to the nearest degree.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

$$= \frac{5-6-2}{3\cdot\sqrt{35}}$$

$$= -\frac{1}{\sqrt{35}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{\sqrt{35}}\right) \approx 99.73^\circ$$

2. (15 points) Find the area of the triangle determined by the points $P(1,-1,2)$, $Q(2,1,3)$, and $R(1,2,-1)$.

$$\vec{PQ} = \langle 1,2,1\rangle \quad \text{and} \quad \vec{PR} = \langle 0,3,-3\rangle$$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 3 & -3 \end{vmatrix}$$

$$= -9\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\text{Area} = \frac{1}{2}\left|\vec{PQ}\times\vec{PR}\right| = \frac{1}{2}\sqrt{81+9+9}$$

$$= \frac{3}{2}\sqrt{11}$$

3. (15 points) Find parametric equations for the line through $(1, 2, 3)$ perpendicular to the plane $4x - y + 7z = 5$, and determine the point where the line intersects the plane.

$$\vec{r}(t) = \langle 1, 2, 3 \rangle + t \langle 4, -1, 7 \rangle$$

$$\Rightarrow \quad x = 1 + 4t, \quad y = 2 - t, \quad z = 3 + 7t$$

Intersects plane when

$$4(1 + 4t) - (2 - t) + 7(3 + 7t) = 5$$

$$\Rightarrow \quad 66t = -18$$

$$\Rightarrow \quad t = -3/11$$

$$\Rightarrow \quad x = -1/11, \quad y = 25/11, \quad z = 12/11$$

so $(-1/11, 25/11, 12/11)$ is the point of intersection

4. (10 points) The position vector of a particle in two dimensions at time t is given by the formula $\mathbf{r}(t) = (t^3 + 1)\mathbf{i} + (20 - 4\sqrt{t + 3})\mathbf{j}$.

(a) Determine the particle's speed at the instant when $t = 1$.

$$\vec{r}\,'(t) = 3t^2\,\hat{\imath} - \frac{2}{\sqrt{t+3}}\,\hat{\jmath}$$

$$\Rightarrow \quad \vec{r}\,'(1) = 3\hat{\imath} - \hat{\jmath}$$

$$\Rightarrow \quad |\vec{r}\,'(1)| = \sqrt{10}$$

(b) Find the x-coordinate of the particle's position at the instant when the y-coordinate is zero.

$$20 - 4\sqrt{t+3} = 0$$

$$\Rightarrow \quad \sqrt{t+3} = 5$$

$$\Rightarrow \quad t = 22$$

so the x-position is $22^3 + 1 = 10649$

5. (10 points) Evaluate $\displaystyle\lim_{\substack{(x,y)\to(3,3)\\ x\neq y}} \frac{x^2-xy}{x^4-y^4}$ or prove that it doesn't exist.

$$= \lim_{\substack{(x,y)\to(3,3)\\ x\neq y}} \frac{x(x-y)}{(x^2-y^2)(x^2+y^2)}$$

$$= \lim_{(x,y)\to(3,3)} \frac{x}{(x+y)(x^2+y^2)}$$

$$= \frac{3}{6\cdot 18}$$

$$= \frac{1}{36}$$

6. (15 points) Consider the function $f(x,y,z) = z - \ln(x^2+y^2)$.

(a) Find the derivative of f at the point $(1,2,3)$ in the direction of $\mathbf{A} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

$$\vec{\nabla} f = -\frac{2x}{x^2+y^2}\,\hat{i} - \frac{2y}{x^2+y^2}\,\hat{j} + \hat{k}$$

$$\vec{\nabla} f\Big|_{(1,2,3)} = -\frac{2}{5}\,\hat{i} - \frac{4}{5}\,\hat{j} + \hat{k}$$

Direction : $\vec{u} = \frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$

$$\vec{\nabla} f\Big|_{(1,2,3)} \cdot \vec{u} = \frac{1}{\sqrt{3}}\left(-\frac{2}{5} + \frac{4}{5} + 1\right) = \frac{7}{5\sqrt{3}}$$

(b) Find an equation for the tangent plane to the level surface of f through the point $(1,2,3)$.

$$-\frac{2}{5}(x-1) - \frac{4}{5}(y-2) + (z-3) = 0$$

$$\Rightarrow \quad -\frac{2}{5}x - \frac{4}{5}y + z = 1$$

$$\Rightarrow \quad 2x + 4y - 5z = -5$$

7. (15 points) Determine the location of all local maxima, local minima, and saddle points of the function
$$f(x, y) = 2x^3 + 3xy + 2y^3.$$

Critical points :

$$\left.\begin{array}{l} f_x = 6x^2 + 3y = 0 \\ f_y = 3x + 6y^2 = 0 \end{array}\right\} \Rightarrow \quad y = -2x^2$$

$$\Rightarrow \quad 3x + 24x^4 = 0$$

$$\Rightarrow \quad 3x(1 + 8x^3) = 0$$

$$\Rightarrow \quad x = 0, \ y = 0$$

$$\text{or} \quad x = -1/2, \ y = -1/2$$

$f_{xx} = 12x < 0$ at $(-1/2, -1/2)$

$f_{yy} = 12y$

$f_{xy} = 3$

$$f_{xx} f_{yy} - f_{xy}^2 = 144xy - 9$$

$$\begin{cases} < 0 & \text{at } (0,0) \\ > 0 & \text{at } (-1/2, -1/2) \end{cases}$$

Saddle point at $(0,0)$

Local max at $(-1/2, -1/2)$

8. (10 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x + y^2$ on the circle $x^2 + y^2 = 1$.

Let $g(x,y) = x^2 + y^2 - 1$

$\vec{\nabla} f = \hat{\imath} + 2y\hat{\jmath}$ and $\vec{\nabla} g = 2x\hat{\imath} + 2y\hat{\jmath}$

So $\vec{\nabla} f = \lambda \vec{\nabla} g$

$$\Rightarrow \begin{cases} 1 = 2x\lambda \\ 2y = 2y\lambda \end{cases} \Rightarrow \quad \lambda = 1, \ x = 1/2, \ y = \pm\sqrt{3}/2$$

$$\text{or} \quad y = 0, \ x = \pm 1, \ \lambda = \pm 1/2$$

$f(1/2, \sqrt{3}/2) = 5/4$

$f(1/2, -\sqrt{3}/2) = 5/4$

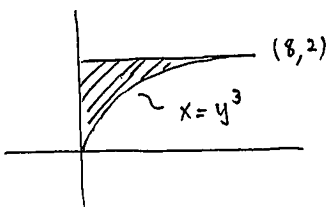
$f(1, 0) = 1$

$f(-1, 0) = -1$

So max is $5/4$

and min is -1

9. (12 points) Evaluate $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$ by reversing the order of integration. Be sure to include a sketch of the region of integration.



$$\int_0^2 \int_0^{y^3} \frac{dx\,dy}{y^4+1}$$

$$= \int_0^2 \frac{y^3}{y^4+1}\,dy$$

$u = y^4 + 1$

$du = 4y^3\,dy$

$$= \frac{1}{4}\int_1^{17} \frac{du}{u}$$

$$= \frac{1}{4}\ln 17$$

10. (13 points) Find the average height of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ above the disk $x^2 + y^2 \le 25$ in the xy-plane.

$$\text{Average height} = \frac{1}{25\pi}\int_0^{2\pi}\int_0^5 \sqrt{25-r^2}\; r\,dr\,d\theta$$

$u = 25 - r^2$

$du = -2r\,dr$

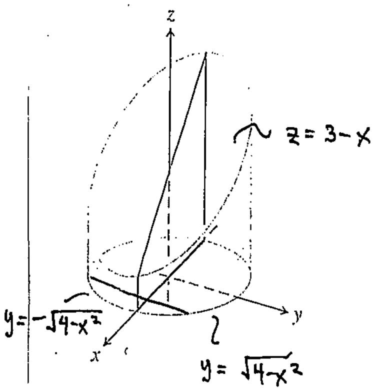
$$= \frac{-1}{50\pi}\int_0^{2\pi}\int_{25}^{0} \sqrt{u}\,du\,d\theta$$

$$= \frac{-1}{50\pi}\int_0^{2\pi} \frac{2}{3}u^{3/2}\Big|_{25}^{0}\,d\theta$$

$$= \frac{1}{50\pi}\cdot 2\pi \cdot \frac{2}{3}\cdot 125$$

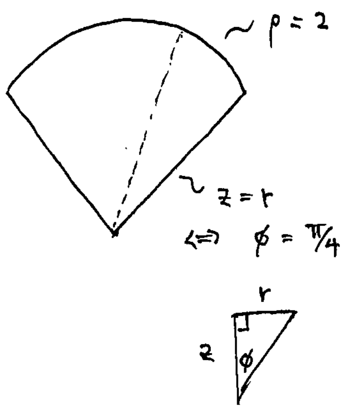
$$= \frac{10}{3}$$

11. (10 points) A solid of density $\delta(x, y, z) = x + y + 5$ and total mass M occupies the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$. Set up (but do not evaluate) an integral that gives the z-coordinate of the object's center of mass.



$$\bar{z} = \frac{1}{M} \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{3-x} z\,(x+y+5)\, dz\, dy\, dx$$

12. (15 points) Find the volume of the ice cream cone bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the cone $z = \sqrt{x^2 + y^2}$.



$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^3}{3}\Big|_0^2 \sin\phi\, d\phi\, d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin\phi\, d\phi\, d\theta$$

$$= -\frac{8}{3} \int_0^{2\pi} \cos\phi \Big|_0^{\pi/4} d\theta$$

$$= -\frac{8}{3} \cdot 2\pi \left(\frac{\sqrt{2}}{2} - 1 \right)$$

$$= \frac{16\pi}{3} \left(1 - \frac{\sqrt{2}}{2} \right)$$

13. (12 points) Find the mass of a wire lying along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 1$, if the density is $\delta(t) = 2t$.

$$\vec{r}'(t) = -\sin t \ \hat{\imath} + \cos t \ \hat{\jmath} + 2t \ \hat{k}$$

$$\Rightarrow \quad |\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4t^2}$$

$$= \sqrt{1 + 4t^2}$$

$$\text{Mass} = \int_0^1 2t \sqrt{1+4t^2} \, dt \qquad u = 1+4t^2, \quad du = 8t \, dt$$

$$= \int_1^5 \frac{1}{4} \sqrt{u} \, du$$

$$= \frac{1}{6} u^{3/2} \Big|_1^5$$

$$= \frac{1}{6} (5\sqrt{5} - 1)$$

14. (13 points) Find the work done by the force $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ over the curve defined by $\mathbf{r}(t) = 4t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}$, $0 \le t \le 1$.

$$\vec{r}'(t) = 4\hat{\imath} + \hat{\jmath} + e^t \hat{k}$$

$$\vec{F}(4t, t, e^t) = e^{2t} \hat{\imath} + 4t \hat{\jmath} + t \hat{k}$$

$$\text{Work} = \int_0^1 (4e^{2t} + 4t + te^t) \, dt$$

IBP

$$u = t \quad dv = e^t$$
$$du = dt \quad v = e^t$$

$$= 2e^{2t} + 2t^2 + te^t - e^t \Big|_0^1$$

$$= 2e^2 + 2 + e - e - (2 - 1)$$

$$= 2e^2 + 1$$

15. (10 points) Find a potential function for the conservative vector field

$$\mathbf{F} = (2xz\sin y + e^x \cos z)\mathbf{i} + (3y^2 + x^2 z\cos y)\mathbf{j} + (x^2 \sin y - e^x \sin z + 4)\mathbf{k}.$$

$$f_x = 2xz\sin y + e^x \cos z \quad \Rightarrow \quad f = x^2 z \sin y + e^x \cos z + g(y,z)$$

$$\Rightarrow \quad f_y = x^2 z \cos y + g_y = 3y^2 + x^2 z \cos y$$

$$\Rightarrow \quad g_y = 3y^2 \quad \Rightarrow \quad g = y^3 + h(z)$$

$$\Rightarrow \quad f = x^2 z \sin y + e^x \cos z + y^3 + h(z)$$

$$\Rightarrow \quad f_z = x^2 \sin y - e^x \sin z + h'(z) = x^2 \sin y - e^x \sin z + 4$$

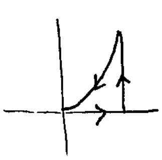
$$\Rightarrow \quad h'(z) = 4 \quad \Rightarrow \quad h(z) = 4z + C$$

Thus $f(x,y,z) = x^2 z \sin y + e^x \cos z + y^3 + 4z + C$

is a potential.

16. (15 points) Suppose that $\mathbf{F} = x^3y^2\mathbf{i} + (x - 2y)\mathbf{j}$ represents the velocity field of a fluid, and let C be the boundary of the region enclosed by the curve $y = x^2$ and the lines $y = 0$ and $x = 1$.

(a) Find the counterclockwise circulation of $\mathbf{F}$ around C.



$$\text{Circulation} = \int_0^1 \int_0^{x^2} (1 - 2x^3y)\, dy\, dx$$

$$= \int_0^1 (y - x^3y^2) \Big|_0^{x^2} dx$$

$$= \int_0^1 (x^2 - x^7)\, dx$$

$$= \frac{x^3}{3} - \frac{x^8}{8} \Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$$

(b) Find the outward flux of $\mathbf{F}$ across C.

$$\text{Flux} = \int_0^1 \int_0^{x^2} (3x^2y^2 - 2)\, dy\, dx$$

$$= \int_0^1 (x^2y^3 - 2y) \Big|_0^{x^2} dx$$

$$= \int_0^1 (x^8 - 2x^2)\, dx$$

$$= \frac{x^9}{9} - \frac{2x^3}{3} \Big|_0^1$$

$$= \frac{1}{9} - \frac{2}{3} = -\frac{5}{9}$$