Final Exam A

1) Let $u = \ln x$, $dv = x^5\,dx$

$du = \frac{1}{x}\,dx$, $v = \frac{1}{6}x^6$

$$\int x^5 \ln x\,dx = \frac{1}{6}x^6 \ln x - \int \frac{1}{6}x^5\,dx$$

$$= \frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C$$

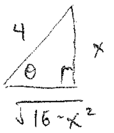
2) Let $x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$\sqrt{16-x^2} = 4\cos\theta$

$$\int \frac{dx}{x\sqrt{16-x^2}} = \int \frac{4\cos\theta\,d\theta}{4\sin\theta \cdot 4\cos\theta}$$

$$= \frac{1}{4}\int \csc\theta\,d\theta$$



$$= -\frac{1}{4}\ln|\csc\theta + \cot\theta| + C$$

$$= -\frac{1}{4}\ln\left|\frac{4}{x} + \frac{\sqrt{16-x^2}}{x}\right| + C$$

3) $$\frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow \quad 1 = A(x^2+4) + (Bx+C)x$$

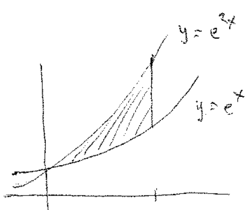
$$= (A+B)x^2 + Cx + 4A$$

$$\Rightarrow \begin{cases} A+B = 0 \\ C = 0 \\ 4A = 1 \end{cases} \Rightarrow A = \frac{1}{4},\quad B = -\frac{1}{4}$$

$$\int \frac{dx}{x^3+4x} = \frac{1}{4}\int\frac{dx}{x} - \frac{1}{4}\int\frac{x\,dx}{x^2+4}$$

$$= \frac{1}{4}\ln|x| - \frac{1}{8}\ln(x^2+4) + C$$

4)

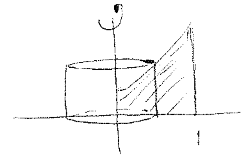


$$A = \int_0^1 (e^{2x} - e^x)\,dx$$

$$= \frac{1}{2}e^{2x} - e^x \Big|_0^1$$

$$= \frac{1}{2}e^2 - e + \frac{1}{2}$$

5)



Shells

$r(x) = x$

$h(x) = x^3 + 1$

$$V = \int_0^1 2\pi x(x^3+1)\,dx$$

$$= 2\pi\int_0^1 (x^4 + x)\,dx$$

$$= 2\pi\left(\frac{x^5}{5} + \frac{x^2}{2}\right)\Big|_0^1$$

$$= \frac{7\pi}{5}$$

6) $$\sum_{n=0}^{\infty} \pi^{-n/2} = 1 + \pi^{-1/2} + \pi^{-1} + \ldots$$

geometric:

$a = 1$

$r = 1/\sqrt{\pi}$

$$= \frac{1}{1 - 1/\sqrt{\pi}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi} - 1}$$

7) Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)^2\,2^{n+1}} \cdot \frac{n^2\,2^n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x-3}{2}\left(\frac{n}{n+1}\right)^2\right|$$

$$= \frac{|x-3|}{2} \qquad \text{so } R = 2$$

Endpts:

$x = 5$ $\quad \sum_{n=1}^{\infty}\frac{1}{n^2}$ convergent p-series

$x = 1$ $\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ abs. conv. by above

So $I = [1, 5]$

8) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{4}{3n^2}\right)$$

$$= \frac{1}{3}$$

Since $\lim_{n\to\infty} a_n \neq 0$,

$\sum_{n=1}^{\infty} a_n$ diverges by the Divergence Test

9) $f(x) = 2x^{3/2} \Rightarrow f'(x) = 3x^{1/2}$

$$L = \int_0^1 \sqrt{1 + (3x^{1/2})^2}\, dx$$

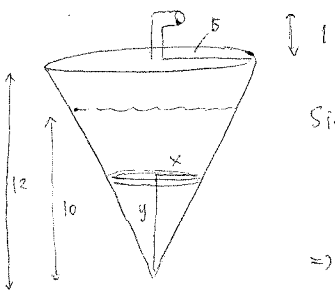
$$= \int_0^1 \sqrt{1+9x}\, dx \qquad \begin{aligned} u &= 1+9x \\ du &= 9\, dx \end{aligned}$$

$$= \int_1^{10} \frac{1}{9}\sqrt{u}\, du$$

$$= \frac{1}{9}\cdot\frac{2}{3} u^{3/2} \Big|_1^{10}$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right)$$

10)



Similar Δ's :

$$\frac{x}{y} = \frac{5}{12}$$

$$\Rightarrow x = \frac{5}{12} y$$

Volume of slice $= \pi x^2\, dy$

$$= \pi \left(\tfrac{5}{12} y\right)^2 dy$$

Weight of slice $= 9800\,\pi \left(\tfrac{5}{12} y\right)^2 dy$

Distance moved by slice $= 13 - y$

$$\text{Work} = \int_0^{10} 9800\,\pi \left(\tfrac{5}{12} y\right)^2 (13 - y)\, dy$$

11)

$$\int_0^\infty \frac{x^2}{(x^3+1)^4}\, dx = \lim_{b\to\infty} \int_0^b \frac{x^2}{(x^3+1)^4}\, dx$$

$u = x^3 + 1$

$du = 3x^2\, dx$

$$= \frac{1}{3} \lim_{b\to\infty} \int_1^{b^3+1} u^{-4}\, du$$

$$= \frac{1}{3} \lim_{b\to\infty} \frac{u^{-3}}{-3} \Big|_1^{b^3+1}$$

$$= -\frac{1}{9} \lim_{b\to\infty} \left(\frac{1}{(b^3+1)^3} - 1 \right)$$

$$= \frac{1}{9}$$

12) $\left| \dfrac{(-1)^n}{\sqrt{n} + 5^n} \right| \le \dfrac{1}{5^n}$ so the series converges absolutely by direct comparison with the geometric series $\sum_{n=1}^{\infty} \frac{1}{5^n}$ $(r = 1/5)$

13)

$f(x) = x^{1/2} \Rightarrow f(1) = 1$

$f'(x) = \frac{1}{2} x^{-1/2} \Rightarrow f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4} x^{-3/2} \Rightarrow f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8} x^{-5/2} \Rightarrow f'''(1) = \frac{3}{8}$

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3$$

$$\sqrt{1.1} \approx T_3(1.1)$$

$$= 1 + \frac{0.1}{2} - \frac{(0.1)^2}{8} + \frac{(0.1)^3}{16}$$

$$= 1.0486875$$

14)

$$\int_0^1 \left(1 - \frac{x^6}{2} + \frac{x^{12}}{24} - \frac{x^{18}}{720} + \cdots\right) dx$$

$$= x - \frac{x^7}{14} + \frac{x^{13}}{13\cdot 24} - \frac{x^{19}}{19\cdot 720} + \cdots \Big|_0^1$$

$$\approx 1 - \frac{1}{14} + \frac{1}{312} \approx 0.9317766$$

Error $\le \dfrac{1}{19\cdot 720}$ by AST

15)

$$\lim_{b\to\infty} \int_3^b \frac{dx}{x(\ln x)(\ln\ln x)^p}$$

$u = \ln\ln x$

$du = \frac{1}{\ln x}\cdot\frac{1}{x}\, dx$

$$= \lim_{b\to\infty} \int_{\ln\ln 3}^{\ln\ln b} \frac{du}{u^p}$$

converges if and only if $\int_1^\infty \frac{1}{u^p}\, du$ converges, which occurs iff $p > 1$.

Final Exam B

1) Let $u = x$, $dv = e^{5x}\,dx$

$du = dx$, $v = \frac{1}{5}e^{5x}$

$$\int xe^{5x}\,dx = \frac{1}{5}xe^{5x} - \int \frac{1}{5}e^{5x}\,dx$$

$$= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C$$

2) Let $x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$\sqrt{9+x^2} = 3\sec\theta$

$$\int \frac{x^3\,dx}{\sqrt{9+x^2}} = \int \frac{27\tan^3\theta \cdot 3\sec^2\theta\,d\theta}{3\sec\theta}$$

$$= 27\int \tan^3\theta\,\sec\theta\,d\theta$$

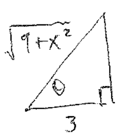
$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$= 27\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$

$$= 27\int (u^2-1)\,du$$

$$= 27\left(\frac{u^3}{3} - u\right) + C$$



$$= 9\sec^3\theta - 27\sec\theta + C$$

$$= \frac{1}{3}(9+x^2)^{3/2} - 9\sqrt{9+x^2} + C$$

3) $$\frac{1}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

$$\Rightarrow \quad 1 = A(x+1) + B(x+3)$$

$$= (A+B)x + A + 3B$$

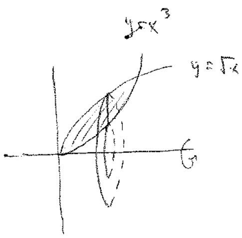
$$\Rightarrow \begin{cases} A+B = 0 \\ A+3B = 1 \end{cases} \Rightarrow A = -\tfrac{1}{2},\ B = \tfrac{1}{2}$$

$$\int \frac{dx}{x^2+4x+3} = -\frac{1}{2}\int \frac{dx}{x+3} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$= -\frac{1}{2}\ln|x+3| + \frac{1}{2}\ln|x+1| + C$$

$$= \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + C$$

4)



Washers: $R(x) = \sqrt{x}$

$r(x) = x^3$

$$V = \int_0^1 \pi(x - x^6)\,dx$$

$$= \pi\left(\frac{x^2}{2} - \frac{x^7}{7}\right)\Big|_0^1 = \frac{5\pi}{14}$$

6) Let $L = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Then $\ln L = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)$

$$= \lim_{n\to\infty}\frac{\ln(1+1/n)}{1/n}$$

$$\underset{L'H}{=} \lim_{n\to\infty}\frac{\frac{1}{1+1/n}\cdot\frac{-1}{n^2}}{-1/n^2}$$

$$= \lim_{n\to\infty}\frac{1}{1+1/n} = 1$$

So $L = e^1 = e$

5) $$\text{av}(f) = \frac{1}{2-0}\int_0^2 x^2\sqrt{x^3+1}\,dx$$

$u = x^3+1$

$du = 3x^2\,dx$

$$= \frac{1}{2}\cdot\frac{1}{3}\int_1^9 \sqrt{u}\,du$$

$$= \frac{1}{6}\cdot\frac{2}{3}u^{3/2}\Big|_1^9$$

$$= \frac{1}{9}(27-1)$$

$$= 26/9$$

7) $$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{5^{n+1}}\cdot\frac{5^n}{(x+2)^n}\right|$$

$$= \frac{|x+2|}{5} \quad \text{so } R = 5$$

$x = 3$: $\sum_{n=1}^{\infty} 1$

$x = -7$: $\sum_{n=1}^{\infty} (-1)^n$

both diverge by Test for Div.

Thus $I = (-7, 3)$

8) False. Consider for example $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$.

Then $\frac{1}{n^2} \le \frac{1}{n}$ and

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges but

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

9) $M_5 = \frac{2}{5}\left[\left(\frac{1}{5}\right)^3 + \left(\frac{3}{5}\right)^3 + 1^3 + \left(\frac{7}{5}\right)^3 + \left(\frac{9}{5}\right)^3\right]$

$= 3.92$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$

so $|f''(x)| \leq 6 \cdot 2 = 12$ on $[0,2]$

Thus $|E_M| \leq \dfrac{12 \cdot 2^3}{24 \cdot 5^2} = 0.16$

10) $A = \int_0^{\pi/2} \cos^3 x \, dx$

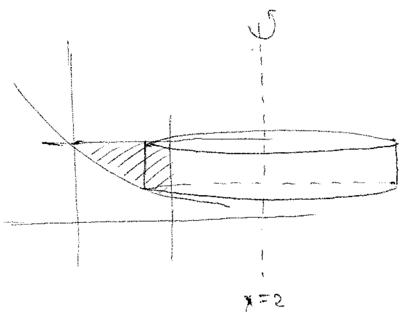
$= \int_0^{\pi/2} \cos x \, (1 - \sin^2 x) \, dx$

$u = \sin x$
$du = \cos x \, dx$

$= \int_0^1 (1 - u^2) \, du$

$= u - \frac{u^3}{3} \Big|_0^1 = \frac{2}{3}$

11)



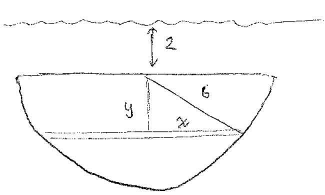
Shells:

$r(x) = 2 - x$

$h(x) = 1 - e^{-x}$

$V = \int_0^1 2\pi (2-x)(1-e^{-x}) \, dx$

12)



$x^2 + y^2 = 36$

$\Rightarrow x = \sqrt{36 - y^2}$

Pressure along strip $= 62.4\,(y+2)$

Area of strip $= 2x \, dy$

$= 2\sqrt{36 - y^2} \, dy$

Force $= \int_0^6 124.8\,(y+2)\sqrt{36-y^2} \, dy$

13) $\int_1^\infty \frac{\tan^{-1} x}{1+x^2} \, dx$

$= \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{1+x^2} \, dx$

$u = \tan^{-1} x$
$du = \frac{1}{1+x^2} \, dx$

$= \lim_{b\to\infty} \int_{\pi/4}^{\tan^{-1} b} u \, du$

$= \lim_{b\to\infty} \frac{1}{2}\left[(\tan^{-1} b)^2 - (\pi/4)^2\right]$

$= \frac{1}{2}\left[(\pi/2)^2 - (\pi/4)^2\right]$

So the series converges absolutely by the Integral Test

14) $\int_0^{1/3} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots\right) dx$

$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots \Big|_0^{1/3}$

$\approx \frac{1}{3} - \frac{1}{3^4} + \frac{1}{3^5 \cdot 10}$

≈ 0.321399

Error $\leq \frac{1}{3^7 \cdot 42}$ by AST

15) $f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2}$

So the object's surface area is

$S = \int_1^\infty 2\pi \cdot \frac{1}{x} \cdot \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \, dx$

$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \, dx$

$\geq 2\pi \int_1^\infty \frac{1}{x} \, dx$

$= 2\pi \lim_{b\to\infty} \ln b$

$= \infty$