IDENTICALLY VANISHING COEFFICIENTS IN THE SERIES EXPANSION OF LACUNARY ETA QUOTIENTS

JAMES MC LAUGHLIN

ABSTRACT. The work described in the present talk had its origins in the following theorem of Han and Ono.

Theorem 0.1. (*Han and Ono 2011*) *Define the sequences* $\{a_n\}$ *and* $\{b_n\}$ *by*

$$f_1^8 =: \sum_{n=0}^{\infty} a_n q^n, \quad \frac{f_3^3}{f_1} =: \sum_{n=0}^{\infty} b_n q^n, \qquad \text{where } f_i := \prod_{n=1}^{\infty} (1-q^{in}), \quad i \in \mathbb{Z}^+. \tag{0.1}$$

Then

$$a_n = 0 \Longleftrightarrow b_n = 0. \tag{0.2}$$

Moreover, we have that $a_n = b_n = 0$ *precisely for those non-negative* n *for which* $ord_p(3n+1)$ *is odd for some prime* $p \equiv 2 \pmod 3$.

In this situation we say that f_1^8 and f_3^3/f_1 have *identically vanishing coefficients*.

Computer investigations by the speaker and his collaborators (Tim Huber (University of Texas, Rio Grande Valley) and Dongxi Ye (Sun Yat-sen University, Guangdong, People's Republic of China) appear to indicate that collections of lacunary eta quotients with identically vanishing coefficients appear to be quite common (for example, experiment suggests that there are at least 70+ eta quotients with coefficients that vanish identically with those f_1^4).

In the present talk we describe what computer experiments appear to suggest, what we have actually been able to prove, and some of the methods that were used to arrive at what has been proved. A number of questions for further investigation are also raised.

Remark: One particular method of proof, m-dissections of eta-quotients, was described in a talk at JMM 2024.

REFERENCES

MATHEMATICS DEPARTMENT, 25 UNIVERSITY AVENUE, WEST CHESTER UNIVERSITY, WEST CHESTER, PA 19383

Email address: jmclaughl@wcupa.edu

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