Vanishing Coefficients in the Series Expansion of Lacunary Eta Quotients

West Chester University Mathematics Colloquium,

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Overview

Background and Notation

q-products

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For $|q| < 1$, $$(q;q)_\infty := (1-q)(1-q^2)(1-q^3)\cdots$$

$$f_1 := (q;q)_\infty \qquad f_j := (q^j;q^j)_\infty$$

The series expansion for f_1:

$$\begin{aligned} f_1 = (q;q)_\infty = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} \\ - q^{35} - q^{40} + q^{51} + q^{57} - q^{70} - q^{77} + q^{92} + q^{100} \\ - q^{117} - q^{126} + q^{145} + q^{155} - q^{176} - q^{187} \dots \end{aligned}$$

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Notice that the coefficients of most powers of q are zero.

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$$1, -1, -1, 0, 0, 1, 0, 1, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0,$$
$$0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
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The series $\sum_{n=0}^{\infty} c(n)q^n$ is *lacunary* if

$$\lim_{x\to\infty} \frac{|\{n \mid 0 \leq n \leq x, c(n)=0\}|}{x} = 1.$$

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One could also ask about more general eta quotients that are lacunary.

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Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $3n + 1$ has a prime factor p of the form $p = 3k + 2$ for some integer k, with odd exponent.

The Result of Han and Ono in More Detail

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$$f_1^8 = 1 - 8q + 20q^2 - 70q^4 + 64q^5 + 56q^6 - 125q^8 - 160q^9 + 308q^{10} + 110q^{12} - 520q^{14} + 57q^{16} + 560q^{17} + 182q^{20} + \dots,$$

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Further, for any n in this list, $3n+1$ has a prime factor p of the form $p = 3k+2$ with odd exponent.

(For example, for $n = 11$, $3n+1 = 3(11)+1 = 34 = 2(17^1)$ and $17 = 3(5)+2$.)

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What was discovered as a result of these computer algebra experiments is summarized as follows.

Other eta quotients with identically vanishing coefficients I

Let $(A(q), B(q))$ be any of the pairs

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}. \quad (3)$$

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For any such pair $(A(q), B(q))$, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a(n) q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b(n) q^n. \qquad (4)$$

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Then, for each pair, $a(n) = 0 \iff b(n) = 0$, with criteria for when exactly this happens (Serre's criteria).

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Later: The results above on identically vanishing coefficients appear to be just "the tip of the iceberg".

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Also, the transformation above takes q^n to q^{3n+1}, and partly explains the relevance of $3n+1$ in the vanishing coefficient criterion.

Interlude: qf_1^{24} and the Ramanujan τ Function

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For example, with $p = 2$ and $r = 3$,
$\tau(2)\tau(2^3) - 2^{11}\tau(2^2) = (-24)84480 - 2^{11}(-1472)$
$= 987136 = \tau(2^4)$.

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As with $\tau(n)$, if $\gcd(m,n) = 1$, then $a_{mn} = a_m a_n$.

Some Sample Proofs

A Proof involving Triangular numbers I

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the set of non-negative integers representable as a sum of two triangular numbers. Thus $b(k)=0$ if and only if $k\notin T_2$.

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However, this is exactly Serre's criterion for $a(n) = 0$.

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Define the sequences $\{a(n)\}$ *and* $\{b(n)\}$ *by*

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Moreover, we have that $a(n) = b(n) = 0$ *precisely for those non-negative* n *for which* $ord_p(6n+1)$ *is odd for some prime* $p \equiv 2 \pmod 3$.

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Remark: For an odd prime p, $p \equiv 2 \pmod 3$ is equivalent to $p \equiv 5 \pmod 6$.

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The next step is to head to the LMFDB (The L-functions and modular forms database (LMFDB)) to look for these CM forms.

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$$+ 48q^{43} - 143q^{49} + 74q^{61} + 32q^{67} + 46q^{73} - 40q^{79} - 176q^{91} - 2q^{97} + \dots$$

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Next, let $S(q)$ denote the CM form of weight 3 and level 144 labelled 144.3.g.c in the LMFDB. Then $S(q)$ has q-series expansion

$$\begin{aligned} S(q) = q &- 8i\sqrt{3}q^7 + 22q^{13} - 16i\sqrt{3}q^{19} - 25q^{25} + 24i\sqrt{3}q^{31} \\ &+ 26q^{37} + 48i\sqrt{3}q^{43} - 143q^{49} + 74q^{61} + 32i\sqrt{3}q^{67} \\ &+ 46q^{73} - 40i\sqrt{3}q^{79} - 176i\sqrt{3}q^{91} - 2q^{97} + \dots. \end{aligned} \tag{8}$$

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The recurrence formula for s_n at prime powers is

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The remainder of the proof is to show that if the factorization of $6n + 1$ is otherwise, then $s_{6n+1} \neq 0$, and hence $b_n \neq 0$.

An Example of the More Usual Kind of Proof VI

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where $\mathcal{N}(\cdot)$ denotes the norm of an ideal.

An Example of the More Usual Kind of Proof VII

In particular, when K is imaginary of discriminant $-d < 0$ and class number 1, In particular, when K is imaginary of discriminant $-d < 0$ and class number 1, one has that $f(z)$ must be a linear combination of the generalized theta series

$$\sum_{\alpha \in \beta + \mathfrak{m}} \alpha^{k-1} q^{\mathcal{N}(\alpha)} \quad \text{over } \beta \in (\mathcal{O}_K/\mathfrak{m})^{\times}$$

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for some integral ideal $\mathfrak{m}$ with $\mathcal{N}(\mathfrak{m}) = N/d$.

An Example of the More Usual Kind of Proof VIII

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Next, following on from the material on the previous slides,

An Example of the More Usual Kind of Proof VIII

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An Example of the More Usual Kind of Proof VIII

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This in turn gives that $b(n) = 0 \iff 6n+1$ has a prime factor $p \equiv 5 \pmod 6$ with odd exponent.

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Define the sequences $\{h_i(n)\}$, $i = 1, \dots, 4$ by

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where H_i are defined several slides back.

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It will be shown that only one of H_1 and H_2 contributes to $s(p)q^p$, and whichever contributes, it contributes exactly two terms.

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If $4|y$, then it can be seen from the exponent of q in the formulae for both H_1 and H_2, that n must be even, since $4m - 2n = y$ or $4m - 2n = -y$.

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An Example of the More Usual Kind of Proof XII

Thus, after simplifying,

$$\begin{aligned}h_2(p) &= \left(-6n+5+(4m_1-2n)\sqrt{-3}\right)^2\\&\quad+\left(-6n+5+(4(n-m_1)-2n)\sqrt{-3}\right)^2\\&= 2\left((-6n+5)^2-3(4m_1-2n)^2\right)=2(x^2-3y^2).\end{aligned}$$

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A similar analysis of the case where H_1 contributes to $s(p)q^p$ when $4|y$, and also of the situation where $4 \not| y$ (whichever of H_1 or H_2 contribute), gives that if $p \equiv 1 \pmod{12}$ is prime, then

$$s(p) = 2(x^2-3y^2) \qquad \text{or} \qquad s(p) = -2(x^2-3y^2).$$

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For our calculations, the key implication in this case ($p \equiv 1 \pmod{12}$) is that,

$$s(p) = \pm 2(x^2 - 3y^2) = \pm 2(x^2 - (p - x^2)) \equiv \pm 4x^2 \pmod{p}$$
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Given what was said earlier, this completes the proof.

Recap I

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Let $(A(q), B(q))$ be any of the pairs

$$\left\{\left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \right.$$
$$\left. \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}. \quad (14)$$

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For any such pair $(A(q), B(q))$, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a(n)q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b(n)q^n. \quad (15)$$

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Then, for each pair, $a(n) = 0 \iff b(n) = 0$, with criteria for when exactly this happens (Serre's criteria).

Recap II

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For the pairs

$$\left\{ \left(f_1^{26}, \frac{f_3^9}{f_1} \right), \left(f_1^{26}, \frac{f_2^{16}}{f_1^6} \right) \right\} \tag{16}$$

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$a(n) = b(n) = 0$ if $12n + 13$ satisfies a criteria of Serre for $a(n) = 0$.

How Extensive is this Phenomenon?

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Notice that each of the triples

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Q. How extensive is this phenomenon of eta quotients with identically vanishing coefficients?

How Extensive is this Phenomenon?

Notice that each of the triples

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}, \frac{f_1^{10}}{f_3^2} \right), \left(f_1^6, \frac{f_2^4}{f_1^2}, \frac{f_1^{14}}{f_2^4} \right), \left(f_1^{14}, \frac{f_3^5}{f_1}, \frac{f_2^8}{f_1^2} \right), \left(f_1^{26}, \frac{f_3^9}{f_1}, \frac{f_2^{16}}{f_1^6} \right) \right\} \quad (17)$$

have identically vanishing coefficients.

Q. How extensive is this phenomenon of eta quotients with identically vanishing coefficients?

A. Quite extensive.

Further Investigations

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Further, in each case there were also many other eta quotients $C(q)$ such that $A_{(0)} \subsetneq C_{(0)}$.

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Further, in each case there were also many other eta quotients $C(q)$ such that $A_{(0)} \subsetneqq C_{(0)}$.

We describe what was found in some detail in the case of f_1^4 and f_1^6.

The Case of f_1^4 I

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In addition, this search found 78 additional eta quotients with the property that for each such eta quotient $C(q)$, it seemed ${f_1^4}_{(0)} \subsetneq C_{(0)}$.

Moreover, it appears that all 150 eta quotients $B(q)$ may be organized into 19 collections (labelled I - XIX in what follows) in a tree-like structure by partially ordering the corresponding $B_{(0)}$ by inclusion.

The Case of f_1^4 II

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Table 1: Eta quotients with vanishing behaviour similar to f_1^4

Collection	# of eta quotients	Collection	# of eta quotients
I	72	II *	4
III †	2	IV	6
V †	2	VI *	4
VII *	6	VIII *	8
IX *	4	X	4
XI	14	XII †	2
XIII †	2	XIV †	2
XV	4	XVI †	2
XVII	4	XVIII †	2
XIX †	6		

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$$XI = \left\{ \frac{f_2 f_8^{14} f_{12}^2}{f_4^6 f_6 f_{16}^5 f_{24}}, \frac{f_6 f_8^{13}}{f_2 f_4^3 f_{12} f_{16}^5}, \frac{f_2^2 f_8 f_{12}^2}{f_4^2 f_{24}}, \frac{f_8^{11}}{f_2^2 f_{16}^5}, \frac{f_4^4 f_{12}^2}{f_2^2 f_8 f_{24}}, \frac{f_2^2 f_8^{13}}{f_4^6 f_{16}^5}, \frac{f_4^{15} f_6 f_{24}}{f_2^5 f_8^5 f_{12}^3}, \frac{f_2^5}{f_6}, \frac{f_2^2 f_4^4}{f_8^2}, \frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}, \frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}, \frac{f_4^{10}}{f_2^2 f_8^4}, \frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}, \frac{f_4^4 f_6^7}{f_2^3 f_{12}^4} \right\}$$

appeared to have identically vanishing coefficients.

The Case of f_1^4 III

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$$XI = \left\{ \frac{f_2 f_8^{14} f_{12}^2}{f_4^6 f_6 f_{16}^5 f_{24}}, \frac{f_6 f_8^{13}}{f_2 f_4^3 f_{12} f_{16}^5}, \frac{f_2^2 f_8 f_{12}^2}{f_4^2 f_{24}}, \frac{f_8^{11}}{f_2^2 f_{16}^5}, \frac{f_4^4 f_{12}^2}{f_2^2 f_8 f_{24}}, \frac{f_2^2 f_8^{13}}{f_4^6 f_{16}^5}, \frac{f_4^{15} f_6 f_{24}}{f_2^5 f_8^5 f_{12}^3}, \frac{f_2^5}{f_6}, \frac{f_2^2 f_4^4}{f_8^2}, \frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}, \frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}, \frac{f_4^{10}}{f_2^2 f_8^4}, \frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}, \frac{f_4^4 f_6^7}{f_2^3 f_{12}^4} \right\}$$

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Collection I is the collection containing f_1^4.

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$$\left. \frac{f_2^5}{f_6}, \frac{f_2^2 f_4^4}{f_8^2}, \frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}, \frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}, \frac{f_4^{10}}{f_2^2 f_8^4}, \frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}, \frac{f_4^4 f_6^7}{f_2^3 f_{12}^4} \right\}$$

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$$\left. \frac{f_2^5}{f_6}, \frac{f_2^2 f_4^4}{f_8^2}, \frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}, \frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}, \frac{f_4^{10}}{f_2^2 f_8^4}, \frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}, \frac{f_4^4 f_6^7}{f_2^3 f_{12}^4} \right\}$$

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$$\left. \frac{f_2^5}{f_6}, \frac{f_2^2 f_4^4}{f_8^2}, \frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}, \frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}, \frac{f_4^{10}}{f_2^2 f_8^4}, \frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}, \frac{f_4^4 f_6^7}{f_2^3 f_{12}^4} \right\}$$

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The relationships between eta quotients in different collections is illustrated in Figure 1.

The Case of f_1^4 IV

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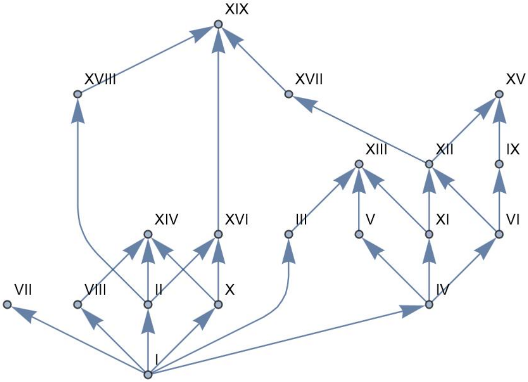

Figure: The grouping of the 150 eta-quotients in Table 1, which have vanishing coefficient behaviour similar to f_1^4

The Case of f_1^4 V

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$$VIII = \left\{ \frac{f_2^3 f_3 f_8 f_{12}^8}{f_1 f_4^3 f_6^4 f_{24}^3}, \frac{f_1 f_8 f_{12}^7}{f_3 f_4^2 f_6 f_{24}^3}, \frac{f_1 f_4^8 f_6^3 f_{24}}{f_2^4 f_3 f_8^3 f_{12}^3}, \frac{f_3 f_4^7 f_{24}}{f_1 f_2 f_8^3 f_{12}^2}, \frac{f_1 f_2^2 f_6}{f_3 f_4}, \frac{f_2^5 f_3 f_{12}}{f_1 f_4^2 f_6^2}, \frac{f_1 f_4 f_6^5}{f_2^2 f_3 f_{12}^2}, \frac{f_2 f_3 f_6^2}{f_1 f_{12}} \right\},$$

$$XIV = \left\{ \frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}, \frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}} \right\},$$

Thus the arrow from VIII to XIV indicates that if $A(q)$ is any of the 8 eta quotients in collection VIII and $B(q)$ is either of the 2 eta quotients in collection XIV, where

$$VIII = \left\{ \frac{f_2^3 f_3 f_8 f_{12}^8}{f_1 f_4^3 f_6^4 f_{24}^3}, \frac{f_1 f_8 f_{12}^7}{f_3 f_4^2 f_6 f_{24}^3}, \frac{f_1 f_4^8 f_6^3 f_{24}}{f_2^4 f_3 f_8^3 f_{12}^3}, \frac{f_3 f_4^7 f_{24}}{f_1 f_2 f_8^3 f_{12}^2}, \frac{f_1 f_2^2 f_6}{f_3 f_4}, \frac{f_2^5 f_3 f_{12}}{f_1 f_4^2 f_6^2}, \frac{f_1 f_4 f_6^5}{f_2^2 f_3 f_{12}^2}, \frac{f_2 f_3 f_6^2}{f_1 f_{12}} \right\},$$

$$XIV = \left\{ \frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}, \frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}} \right\},$$

then $A_{(0)} \subsetneq B_{(0)}$.

The Case of f_1^4 V

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$$XIV = \left\{ \frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}, \frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}} \right\},$$

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A similar meaning for any other arrow in this figure is to be understood.

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then $A_{(0)} \subsetneq B_{(0)}$.

A similar meaning for any other arrow in this figure is to be understood.

The inclusion just mentioned, between groups VIII and XIV, is one of several such inclusion results indicated by the arrows in Figure 1 that have been proven.

The Case of f_1^6 I

The Case of f_1^6 I

Table 2: Eta quotients with vanishing behaviour similar to f_1^6

Collection	# of eta quotients	Collection	# of eta quotients
I	42	II *	4
III *	4	IV	16
V †	2	VI †	2
VII *	4	VIII *	4
IX *	4	X	10
XI †	2	XII *	4
XIII *	8	XIV *	4
XV	8	XVI †	2
XVII	8	XVIII †	2
XIX †	2	XX †	2
XXI *	4	XXII *	6
XXIII †	2	XXIV *	4
XXV *	4	XXVI	4
XXVII †	2	XXVIII †	6
XXIX †	6		

The Case of f_1^6 II

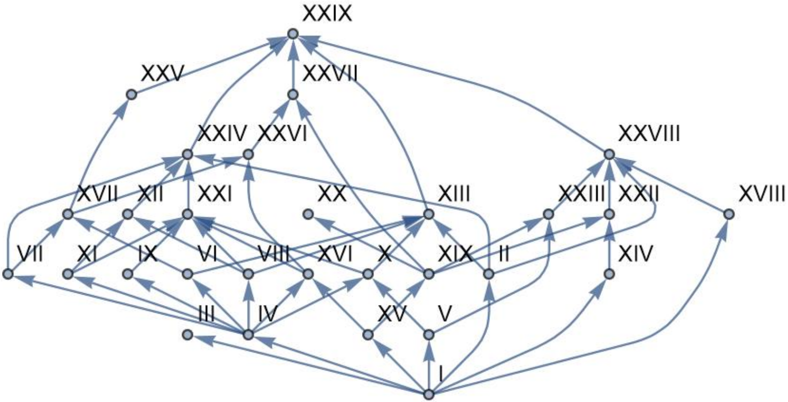

Figure: The grouping of eta-quotients in Table 2, which have vanishing coefficient behaviour similar to f_1^6

The Case of f_1^8 I

The Case of f_1^8 I

Table 3: Eta quotients with vanishing behaviour similar to f_1^8

Collection	# of eta quotients	Collection	# of eta quotients
I	24	II †	2
III †	2	IV	60
V †	2	VI	6
VII †	2	VIII	4
IX †	2	X †	2
XI *	4	XII *	4
XIII *	4	XIV	4
XV †	2	XVI †	2
XVII †	2	XVIII †	2
XIX	6	XX †	2
XXI †	2	XXII †	4
XXIII †	2	XXIV	4
XXV †	6		

The Case of f_1^8 II

The Case of f_1^8 II

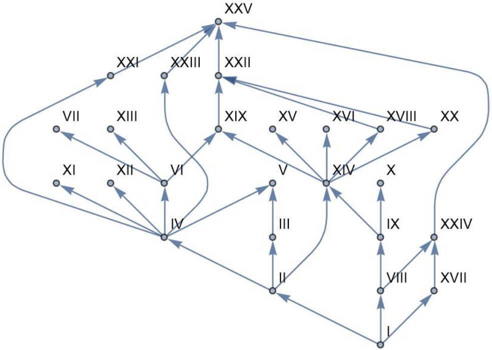

Figure: The grouping of eta-quotients in Table 3, which have vanishing coefficient behaviour similar to f_1^8

The Case of f_1^8 II

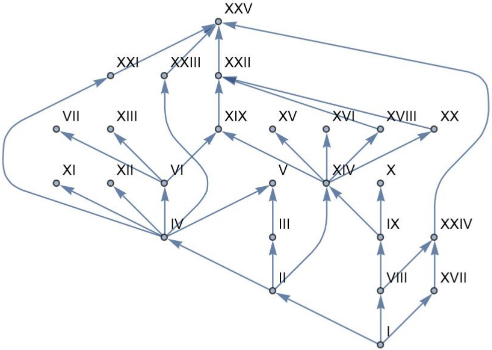

Figure: The grouping of eta-quotients in Table 3, which have vanishing coefficient behaviour similar to f_1^8

Remark: If the tables and graphs represent the true situation for f_1^4 and f_1^8,

The Case of f_1^8 II

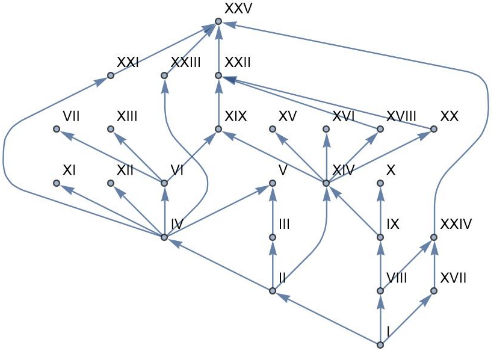

Figure: The grouping of eta-quotients in Table 3, which have vanishing coefficient behaviour similar to f_1^8

Remark: If the tables and graphs represent the true situation for f_1^4 and f_1^8, then the entire table and graph for f_1^4 is embedded in those for f_1^8 via a $q \to q^2$ dilation.

The Case of f_1^8 II

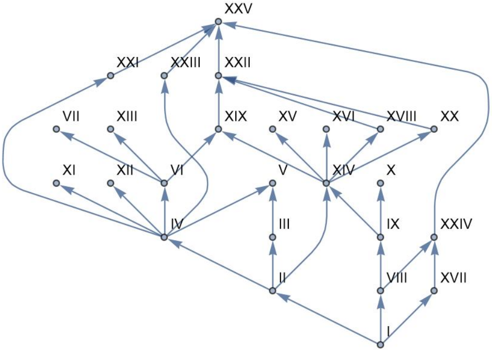

Figure: The grouping of eta-quotients in Table 3, which have vanishing coefficient behaviour similar to f_1^8

Remark: If the tables and graphs represent the true situation for f_1^4 and f_1^8, then the entire table and graph for f_1^4 is embedded in those for f_1^8 via a $q \to q^2$ dilation.

The Case of f_1^{10} I

The Case of f_1^{10} I

Table 4: Eta quotients with vanishing behaviour similar to f_1^{10}

Collection	# of eta quotients	Collection	# of eta quotients
I	38	II *	4
III †	2	IV *	4
V	4	VI †	2
VII	6	VIII †	2
IX *	4	X †	2
XI *	4	XII †	2
XIII †	2	XIV †	2
XV †	2	XVI †	2
XVII	8	XVIII †	2
XIX *	4	XX †	2
XXI †	2	XXII †	2
XXIII	4	XXIV †	4
XXV †	6		

The Case of f_1^{10} II

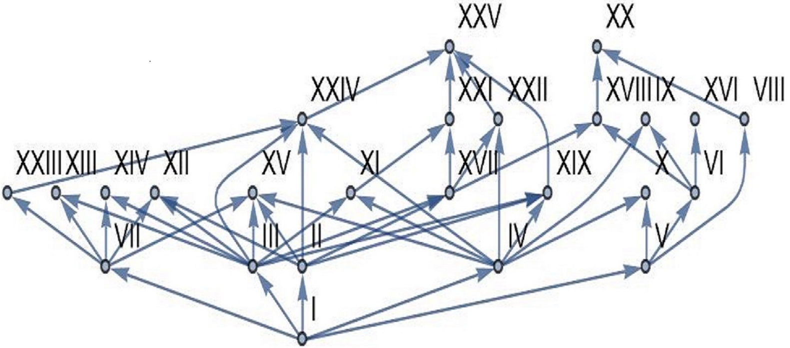

Figure: The grouping of eta-quotients in Table 4, which have vanishing coefficient behaviour similar to f_1^{10}

The Case of f_1^{14} I

The Case of f_1^{14} I

Table 5: Eta quotients with vanishing behaviour similar to f_1^{14}

Collection	# of eta quotients	Collection	# of eta quotients
I	32	II *	4
III *	4	IV *	4
V †	2	VI	12
VII *	4	VIII	8
IX †	2	X †	2
XI †	2	XII †	2
XIII †	2	XIV †	4
XV †	6		

The Case of f_1^{14} II

The Case of f_1^{14} II

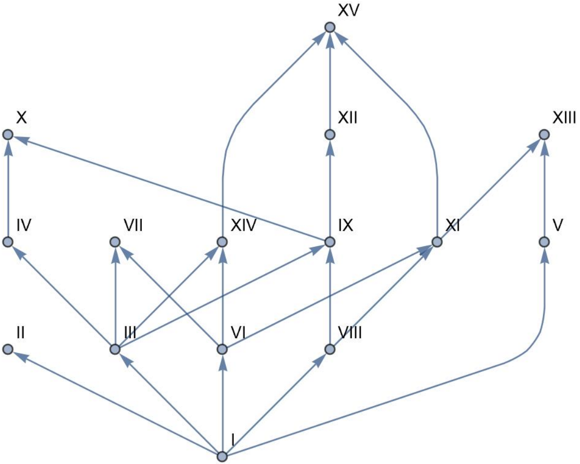

Figure: The grouping of eta-quotients in Table 5, which have vanishing coefficient behaviour similar to f_1^{14}

The Case of f_1^{26} I

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Table 6: Eta quotients with vanishing behaviour similar to f_1^{26}

Collection	# of eta quotients	Collection	# of eta quotients
I	12	II	4
III *	4	IV †	2
V †	2	VI †	2
VII †	2	VIII	4
IX	8	X †	2
XI	8	XII †	2
XIII	12	XIV	10
XV †	2	XVI †	2
XVII †	4	XVIII †	6

The Case of f_1^{26} II

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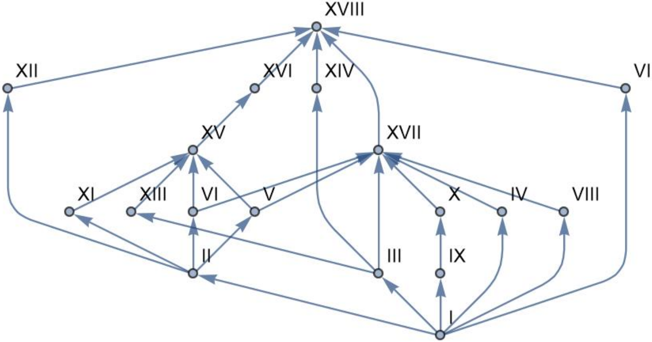

Figure: The grouping of eta-quotients in Table 6, which have vanishing coefficient behaviour similar to f_1^{26}

The Case of $f_1^3 f_2^3$ I

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Table 7: Eta quotients with vanishing behaviour similar to $f_1^3 f_2^3$

Collection	# of eta quotients	Collection	# of eta quotients
I	40	II *	6
III †	2	IV †	2
V †	2	VI †	2
VII †	2	VIII	8
IX	14	X †	2
XI *	4	XII *	4
XIII	10	XIV †	2
XV †	2	XVI †	2
XVII †	6	XVIII †	6

The Case of $f_1^3 f_2^3$ II

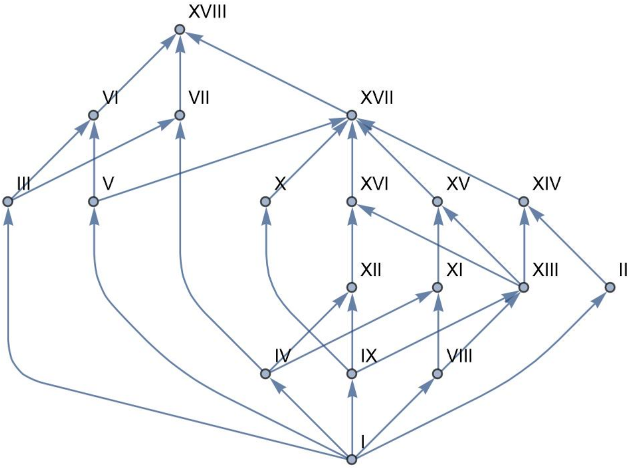

Figure: The grouping of eta-quotients in Table 7, which have vanishing coefficient behaviour similar to $f_1^3 f_2^3$

General Inclusion Results

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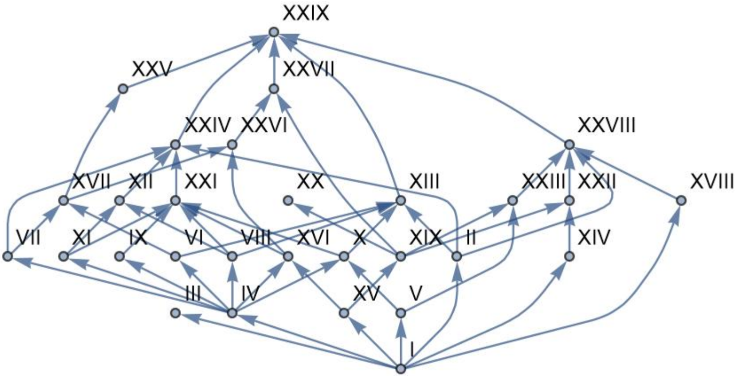

Figure: The grouping of the 172 eta-quotients in Table 2, which have vanishing coefficient behaviour similar to f_1^6

General Inclusion Results III

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$$f_1^6 = \sum_{n=0}^{\infty} a_n q^n, \tag{18}$$

one has that $a_n = 0$ if and only if $4n + 1$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent.

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Remark: All the work in finding representations of eta quotients in the tables as products of two eta quotients with theta series expansions was performed by *Mathematica*.

General Inclusion Results VII

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General Inclusion Results VIII

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$$h_2(q;j,k)=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} q^{(24m+j)^2+4(24n+k)^2},$$

$$g_1(q;j,k)=\sum_{m,n=0}^{\infty} q^{(20m+j)^2+(20n+k)^2},$$

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Then $qB(q^4)$ is a linear combination of $h_i(q;j,k)$ for $i \in \{1,2\}$ and $0 \leq j,k \leq 23$ and $g_i(q;,j,k)$ for $i \in \{1,2\}$ and $0 \leq j,k \leq 19$.

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Dissection Methods

Recap I

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Recall:

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Table 8: Eta quotients with vanishing behaviour similar to f_1^4

Collection	# of eta quotients	Collection	# of eta quotients
I	72	II *	4
III †	2	IV	6
V †	2	VI *	4
VII *	6	VIII *	8
IX *	4	X	4
XI	14	XII †	2
XIII †	2	XIV †	2
XV	4	XVI †	2
XVII	4	XVIII †	2
XIX †	6		

Recap II

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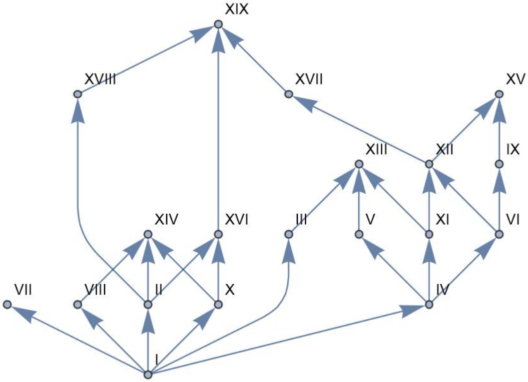

Figure: The grouping of the 150 eta-quotients in Table 1, which have vanishing coefficient behaviour similar to f_1^4

Recap III

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We next describe a method that allows some of this fine structure to be proven.

The m-Dissection of a Function,I

Definition

By the *m-dissection* of a function $G(q)=\sum_{n=0}^{\infty} g_n q^n$ we mean an expansion of the form

$$G(q)=\gamma_0 G_0(q^m)+\gamma_1 q G_1(q^m)+\cdots+\gamma_{m-1}q^{m-1}G_{m-1}(q^m), \qquad (19)$$

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where each dissection component $G_i(q^m)$ is not identically zero ($\gamma_i=0$ is allowed). In other words, for each i, $0\leq i\leq m-1$,

$$\gamma_i\, q^i\, G_i(q^m)=\sum_{n=0}^{\infty} g_{mn+i}q^{mn+i}=q^i\sum_{n=0}^{\infty} g_{mn+i}(q^m)^n.$$

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Now suppose $C(q)$ and $D(q)$ are two functions whose m-dissections are given by

$$C(q) = c_0 G_0(q^m) + c_1 q G_1(q^m) + \cdots + c_{m-1} q^{m-1} G_{m-1}(q^m), \qquad (20)$$
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By splitting the series expansion of an eta quotient into sub-series over arithmetic progression, it may be possible to derive an m-dissection in terms of infinite products.

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The relevance in the present context is that a function and its $q \to -q$ partner have identically vanishing coefficients.

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Specializing $C(q^2)$ then shows that various collections of 4 eta quotients in some of the tables have identically vanishing coefficients.

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$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \implies \frac{f_6}{f_3}\frac{f_1^2}{f_2} = \frac{f_6 f_9^2}{f_3 f_{18}} - 2q\frac{f_{18}^2}{f_9} \tag{43}$$

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_{18}^5}{f_9^2 f_{36}^2} + \frac{2q f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}}, \tag{44}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{45}$$

$$\frac{f_1^2 f_4^2}{f_2^5} = \frac{f_3^8 f_{12}^8 f_{18}^{15}}{f_6^{20} f_9^6 f_{36}^6} - \frac{2q f_3^7 f_{12}^7 f_{18}^9}{f_6^{18} f_9^3 f_{36}^3} + \frac{4q^2 f_3^6 f_{12}^6 f_{18}^3}{f_6^{16}}. \tag{46}$$

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The following 3-dissections are also well known:

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{41}$$

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Aside: The functions above satisfy the identity

$$a(q)^3 = b(q)^3 + c(q)^3.$$

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$$f_1^3 = a(q^3)f_3 - 3qf_9^3, \tag{48}$$

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More Vanishing Coefficient Results, II

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As with the previous theorem, here also Specializing $C(q^3)$ then shows that various collections of 4 eta quotients in some of the tables have identically vanishing coefficients.

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Recall

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We will use the second identity with $q \to q^2$.

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Proof.

For (56), write

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and use (54) and (55), with q replaced with q^2 in the latter identity.

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Some 4-Dissections, IV

Apart from the known dissections, the new dissection identities were motivated by computer searches that went through the various tables of eta quotients and looked for pairs of eta quotients that seemed to similar m-dissections, for $m = 2, 3, 4, 5, 6, 7$ and 8.

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These experimental searches did indeed lead to a quite large number of m-dissection identities, which in turned allowed us to prove that certain collections of eta quotients did indeed have identically vanishing coefficients.

All of the new dissection results in the paper were derived to prove similar m-dissection results for pairs of eta quotients that were found experimentally.

More New m-Dissection Results, I

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$$\frac{f_2}{f_1^2} = \frac{f_8^4}{f_4^{10}} \left(\frac{f_8^5}{f_4^2 f_{16}^2} + 2q \frac{f_{16}^2}{f_8} \right) \left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2 \frac{f_4^2 f_{16}^4}{f_8^2} \right), \qquad (60)$$

$$f_1^2 f_2^3 = \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2 \frac{f_4^2 f_{16}^4}{f_8^2} \right) \qquad (61)$$

$$= \frac{f_8^{15}}{f_4^4 f_{16}^6} - \frac{2q f_8^9}{f_4^2 f_{16}^2} - 4q^2 f_8^3 f_{16}^2 + \frac{8q^3 f_4^2 f_{16}^6}{f_8^3}, \qquad (62)$$

$$\frac{f_1^6}{f_2^3} = \frac{f_8^{15}}{f_4^6 f_{16}^6} - 6q \frac{f_8^9}{f_4^4 f_{16}^2} + 12q^2 \frac{f_8^3 f_{16}^2}{f_4^2} - 8q^3 \frac{f_{16}^6}{f_8^3}. \qquad (63)$$

Notice that f_2/f_1^2, $f_1^2 f_2^3 (f_8^4/f_4^{10})$ and $f_1^6 f_8^4 / f_2^3 f_4^8$ have similar 4-dissections,

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More New m-Dissection Results, III

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Let $C(q^4)$ be any eta quotient whose series expansion contains only powers of q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:

$$\left\{\frac{f_2}{f_1^2}C(q^4),\ \frac{f_1^2 f_4^2}{f_2^5}C(q^4),\ \frac{f_1^2 f_2^3 f_8^4}{f_4^{10}}C(q^4),\ \frac{f_2^9 f_8^4}{f_1^2 f_4^{12}}C(q^4),\ \frac{f_1^6 f_8^4}{f_2^3 f_4^8}C(q^4),\ \frac{f_2^{15} f_8^4}{f_1^6 f_4^{14}}C(q^4)\right\}. \quad (64)$$

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$$\left. \frac{f_2^9 f_8^4}{f_1^2 f_4^{12}}C(q^4),\ \frac{f_1^6 f_8^4}{f_2^3 f_4^8}C(q^4),\ \frac{f_2^{15} f_8^4}{f_1^6 f_4^{14}}C(q^4)\right\}. \qquad (64)$$

Then

$$F_{(0)} = G_{(0)}. \qquad (65)$$

Remark: The claim for three of these eta quotients follow from the remarks on the previous slide,

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$$F_{(0)} = G_{(0)}. \quad (65)$$

Remark: The claim for three of these eta quotients follow from the remarks on the previous slide, and the claim for the other three follow, since they are the $q \to -q$ partners of the first three.

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We need the lemma in the next slide.

We recall the notation, for a an integer and m a positive integer,

$$\bar{J}_{a,m} := (-q^a, -q^{m-a}, q^m; q^m)_\infty. \tag{66}$$

More New m-Dissection Results, V

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$$f_1 = \frac{f_2}{f_4}\left(\bar{J}_{6,16} - q\bar{J}_{2,16}\right), \tag{67}$$

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Proof.

The identity (68) was proven by Hirschhorn,

More New m-Dissection Results, V

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Proof.

The identity (68) was proven by Hirschhorn, and (67) is its $q \to -q$ partner. □

The next long list of pairs of 4-dissections is derived by combining the dissections above with the basic 2- and 3- dissections in ways similar to what has been seen already.

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$$\frac{f_1^2}{f_2^2} = \frac{1}{f_4^2}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\bar{J}_{12,32} + q^2\bar{J}_{4,32}\right), \tag{69}$$

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More New m-Dissection Results, VIII

$$\frac{f_3}{f_1} = \frac{f_{12}}{f_4^4}\left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} + q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right)\left(\frac{f_8 f_{32} f_{48}^2}{f_{16} f_{24} f_{96}} + q^2\frac{f_{16}^2 f_{96}}{f_{32} f_{48}}\right)\left(\bar{J}_{12,32} + q^2 \bar{J}_{4,32}\right),$$

$$\frac{f_1 f_2 f_6}{f_3} = \frac{f_4 f_{24}}{f_8^2 f_{12}}\left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} - q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right)\left(\frac{f_8 f_{32} f_{48}^2}{f_{16} f_{24} f_{96}} - q^2\frac{f_{16}^2 f_{96}}{f_{32} f_{48}}\right)\left(\bar{J}_{12,32} - q^2 \bar{J}_{4,32}\right), \tag{73}$$

More New m-Dissection Results, IX

More New m-Dissection Results, IX

$$\frac{f_1}{f_3} = \frac{f_4}{f_{12}^4}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}} - q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\frac{f_{32}f_{48}^2}{f_{16}f_{96}} - q^2\frac{f_{16}^2f_{24}f_{96}}{f_8f_{32}f_{48}}\right)\left(\bar{J}_{36,96} + q^6\bar{J}_{12,96}\right),$$

$$\frac{f_2f_3f_6}{f_1} = \frac{f_8f_{12}}{f_4f_{24}^2}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}} + q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\frac{f_{32}f_{48}^2}{f_{16}f_{96}} + q^2\frac{f_{16}^2f_{24}f_{96}}{f_8f_{32}f_{48}}\right)\left(\bar{J}_{36,96} - q^6\bar{J}_{12,96}\right), \tag{74}$$

More New m-Dissection Results, X

More New m-Dissection Results, X

$$\frac{f_1^2}{f_6^2} = \frac{f_4}{f_{12}^4}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{32} f_{48}^2}{f_{16} f_{96}} - q^2\frac{f_{16}^2 f_{24} f_{96}}{f_8 f_{32} f_{48}}\right)\left(\bar{J}_{36,96} + q^6 \bar{J}_{12,96}\right), \tag{75}$$

$$\frac{f_1^2 f_6^2}{f_2^2} = \frac{f_8 f_{12}^2}{f_4^2 f_{24}^2}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{32} f_{48}^2}{f_{96} f_{16}} + q^2\frac{f_{16}^2 f_{24} f_{96}}{f_8 f_{32} f_{48}}\right)\left(\bar{J}_{36,96} - q^6 \bar{J}_{12,96}\right), \tag{76}$$

More New m-Dissection Results, X

$$\frac{f_1^2}{f_6^2} = \frac{f_4}{f_{12}^4}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{32} f_{48}^2}{f_{16} f_{96}} - q^2\frac{f_{16}^2 f_{24} f_{96}}{f_8 f_{32} f_{48}}\right)\left(\bar{J}_{36,96} + q^6 \bar{J}_{12,96}\right), \quad (75)$$

$$\frac{f_1^2 f_6^2}{f_2^2} = \frac{f_8 f_{12}^2}{f_4^2 f_{24}^2}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{32} f_{48}^2}{f_{96} f_{16}} + q^2\frac{f_{16}^2 f_{24} f_{96}}{f_8 f_{32} f_{48}}\right)\left(\bar{J}_{36,96} - q^6 \bar{J}_{12,96}\right), \quad (76)$$

$$\frac{f_3}{f_1 f_2^3 f_6} = \frac{f_8^4}{f_4^{14}}\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + \frac{4 f_4^2 f_{16}^4 q^2}{f_8^2}\right)\left(\frac{f_4 f_{16} f_{24}^2}{f_{12} f_{48} f_8} + \frac{f_{48} f_8^2 q}{f_{16} f_{24}}\right)\left(J_{12,32} + q^2 J_{4,32}\right), \quad (77)$$

$$\frac{f_2^7 f_3}{f_1 f_6} = \frac{f_4}{f_8}\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - \frac{4 f_4^2 f_{16}^4 q^2}{f_8^2}\right)\left(\frac{f_4 f_{16} f_{24}^2}{f_{12} f_{48} f_8} + \frac{f_{48} f_8^2 q}{f_{16} f_{24}}\right)\left(J_{12,32} - q^2 J_{4,32}\right), \quad (78)$$

More New m-Dissection Results, XI

More New m-Dissection Results, XI

$$\frac{f_2^5 f_3^2}{f_6} = \frac{f_4}{f_8}\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right)\left(\frac{f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32} - q^2 J_{4,32}\right), \tag{79}$$

$$\frac{f_6^5}{f_2^5 f_3^2} = \frac{f_8^4 f_{12}^2}{f_4^{14}}\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right)\left(\frac{f_{24}^5}{f_{12}^2 f_{48}^2} + 2q^3\frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32} + q^2 J_{4,32}\right), \tag{80}$$

More New m-Dissection Results, XI

$$\frac{f_2^5 f_3^2}{f_6} = \frac{f_4}{f_8}\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2 \frac{f_4^2 f_{16}^4}{f_8^2}\right)\left(\frac{f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3 \frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32} - q^2 J_{4,32}\right), \tag{79}$$

$$\frac{f_6^5}{f_2^5 f_3^2} = \frac{f_8^4 f_{12}^2}{f_4^{14}}\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2 \frac{f_4^2 f_{16}^4}{f_8^2}\right)\left(\frac{f_{24}^5}{f_{12}^2 f_{48}^2} + 2q^3 \frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32} + q^2 J_{4,32}\right), \tag{80}$$

$$\frac{f_2 f_3^2}{f_6} = \frac{f_4}{f_8}\left(\frac{f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3 \frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32} - q^2 J_{4,32}\right), \tag{81}$$

$$\frac{f_6^5}{f_2 f_3^2} = \frac{f_{12}^2}{f_4^2}\left(\frac{f_{24}^5}{f_{12}^2 f_{48}^2} + 2q^3 \frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32} + q^2 J_{4,32}\right), \tag{82}$$

More New m-Dissection Results, XII

More New m-Dissection Results, XII

$$\frac{f_1 f_6^2}{f_2^4 f_3} = \frac{1}{f_4^6}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{12,32} + q^2 \bar{J}_{4,32}\right)^3, \tag{83}$$

$$\frac{f_1 f_2^2 f_6^2}{f_3} = \frac{f_4^3}{f_8^3}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{12,32} - q^2 \bar{J}_{4,32}\right)^3, \tag{84}$$

More New m-Dissection Results, XII

$$\frac{f_1 f_6^2}{f_2^4 f_3} = \frac{1}{f_4^6}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{12,32} + q^2 \bar{J}_{4,32}\right)^3, \tag{83}$$

$$\frac{f_1 f_2^2 f_6^2}{f_3} = \frac{f_4^3}{f_8^3}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{12,32} - q^2 \bar{J}_{4,32}\right)^3, \tag{84}$$

$$\frac{f_1^2 f_6}{f_2^3} = \frac{f_{12}}{f_4^4}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8 f_{32} f_{48}^2}{f_{16} f_{24} f_{96}} + q^2\frac{f_{16}^2 f_{96}}{f_{32} f_{48}}\right)\left(\bar{J}_{12,32} + q^2 \bar{J}_{4,32}\right), \tag{85}$$

$$\frac{f_1^2 f_2}{f_6} = \frac{f_4^2 f_{24}}{f_8^2 f_{12}^2}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8 f_{32} f_{48}^2}{f_{16} f_{24} f_{96}} - q^2\frac{f_{16}^2 f_{96}}{f_{32} f_{48}}\right)\left(\bar{J}_{12,32} - q^2 \bar{J}_{4,32}\right), \tag{86}$$

More New m-Dissection Results, XIII

More New m-Dissection Results, XIII

$$\frac{f_1 f_6^2}{f_3} = \frac{f_4}{f_8}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{12,32} - q^2 \bar{J}_{4,32}\right), \tag{87}$$

$$\frac{f_1 f_6^2}{f_2^2 f_3} = \frac{1}{f_4^2}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{12,32} + q^2 \bar{J}_{4,32}\right), \tag{88}$$

More New m-Dissection Results, XIII

$$\frac{f_1 f_6^2}{f_3} = \frac{f_4}{f_8}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{12,32} - q^2 \bar{J}_{4,32}\right), \tag{87}$$

$$\frac{f_1 f_6^2}{f_2^2 f_3} = \frac{1}{f_4^2}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{12,32} + q^2 \bar{J}_{4,32}\right), \tag{88}$$

$$\begin{aligned}\frac{f_2^2 f_3}{f_1 f_6^6} &= \frac{f_4^2 f_{24}^6}{f_8^2 f_{12}^{17}}\left(\frac{f_8^3 f_{12}^2}{f_4^2 f_{24}} - q^2\frac{f_{24}^3}{f_8}\right)^2 \\ &\quad\times\left(\frac{f_4 f_{16} f_{24}^2}{f_{12} f_{48} f_8} + q\frac{f_{48} f_8^2}{f_{16} f_{24}}\right)\left(\frac{f_8 f_{32} f_{48}^2}{f_{24} f_{96} f_{16}} + q^2\frac{f_{96} f_{16}^2}{f_{32} f_{48}}\right)\left(\bar{J}_{12,32} + q^2 \bar{J}_{4,32}\right), \\ \frac{f_1 f_6^7}{f_2 f_3} &= \frac{f_4 f_{24}}{f_8^2 f_{12}}\left(\frac{f_{12}^2 f_8^3}{f_4^2 f_{24}} + q^2\frac{f_{24}^3}{f_8}\right)^2 \\ &\times\left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} - q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right)\left(\frac{f_8 f_{32} f_{48}^2}{f_{16} f_{24} f_{96}} - q^2\frac{f_{16}^2 f_{96}}{f_{32} f_{48}}\right)\left(\bar{J}_{12,32} - q^2 \bar{J}_{4,32}\right),\end{aligned} \tag{89}$$

More New m-Dissection Results, XIV

$$\frac{f_1^2}{f_2 f_6^5} = \frac{f_{24}^4}{f_{12}^{14}} \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\frac{f_{24}^{10}}{f_{12}^2 f_{48}^4} + 4q^6 \frac{f_{12}^2 f_{48}^4}{f_{24}^2} \right) \left(\bar{J}_{36,96} + q^6 \bar{J}_{12,96} \right), \tag{90}$$

$$\frac{f_1^2 f_6^5}{f_2} = \frac{f_{12}}{f_{24}} \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\frac{f_{24}^{10}}{f_{12}^2 f_{48}^4} - 4q^6 \frac{f_{12}^2 f_{48}^4}{f_{24}^2} \right) \left(\bar{J}_{36,96} - q^6 \bar{J}_{12,96} \right), \tag{91}$$

More New m-Dissection Results, XIV

$$\frac{f_1^2}{f_2 f_6^5} = \frac{f_{24}^4}{f_{12}^{14}} \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\frac{f_{24}^{10}}{f_{12}^2 f_{48}^4} + 4q^6 \frac{f_{12}^2 f_{48}^4}{f_{24}^2} \right) \left(\bar{J}_{36,96} + q^6 \bar{J}_{12,96} \right), \tag{90}$$

$$\frac{f_1^2 f_6^5}{f_2} = \frac{f_{12}}{f_{24}} \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\frac{f_{24}^{10}}{f_{12}^2 f_{48}^4} - 4q^6 \frac{f_{12}^2 f_{48}^4}{f_{24}^2} \right) \left(\bar{J}_{36,96} - q^6 \bar{J}_{12,96} \right), \tag{91}$$

$$\frac{f_1^2}{f_2 f_6} = \frac{1}{f_{12}^2} \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\bar{J}_{36,96} + q^6 \bar{J}_{12,96} \right), \tag{92}$$

$$\frac{f_1^2 f_6}{f_2} = \frac{f_{12}}{f_{24}} \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\bar{J}_{36,96} - q^6 \bar{J}_{12,96} \right), \tag{93}$$

More New m-Dissection Results, XV

More New m-Dissection Results, XV

$$\frac{f_1 f_6}{f_2 f_3} = \frac{1}{f_{12}^2}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{36,96} + q^6 \bar{J}_{12,96}\right), \tag{94}$$

$$\frac{f_1 f_6^3}{f_2 f_3} = \frac{f_{12}}{f_{24}}\left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - q\frac{f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)\left(\bar{J}_{36,96} - q^6 \bar{J}_{12,96}\right). \tag{95}$$

More New m-Dissection Results, XVI

More New m-Dissection Results, XVI

The 4-dissections above lead to the following theorem on collections of eta quotients with identically vanishing coefficients.

More New m-Dissection Results, XVI

The 4-dissections above lead to the following theorem on collections of eta quotients with identically vanishing coefficients.

Theorem. *Let* $C(q^4)$ *be any eta quotient with a power series expansion in* q^4*.*

The 4-dissections above lead to the following theorem on collections of eta quotients with identically vanishing coefficients.

Theorem. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients from one the following lists:*

More New m-Dissection Results, XVI

The 4-dissections above lead to the following theorem on collections of eta quotients with identically vanishing coefficients.

Theorem. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients from one the following lists:*

$$\left\{ f_1^2 C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^3}{f_2^2 f_8} C\left(q^4\right), \frac{f_2^4 f_4}{f_1^2 f_8} C\left(q^4\right) \right\}, \tag{96}$$

$$\left\{ f_1^2 f_2^2 C\left(q^4\right), \frac{f_2^8}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^9}{f_2^4 f_8^3} C\left(q^4\right), \frac{f_2^2 f_4^7}{f_1^2 f_8^3} C\left(q^4\right) \right\}, \tag{97}$$

$$\left\{ \frac{f_3}{f_1} C\left(q^4\right), \frac{f_1 f_4 f_6^3}{f_2^3 f_3 f_{12}} C\left(q^4\right), \frac{f_1 f_2 f_6 f_8^2 f_{12}^2}{f_3 f_4^5 f_{24}} C\left(q^4\right), \frac{f_2^4 f_3 f_8^2 f_{12}^3}{f_1 f_4^6 f_6^2 f_{24}} C\left(q^4\right). \right\}, \tag{98}$$

$$\left\{ \frac{f_1}{f_3} C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6^3} C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_6 f_{24}^2}{f_1 f_8 f_{12}^5} C\left(q^4\right), \frac{f_1 f_4^3 f_6^4 f_{24}^2}{f_2^2 f_3 f_8 f_{12}^6} C\left(q^4\right) \right\}, \tag{99}$$

More New m-Dissection Results, XVI

The 4-dissections above lead to the following theorem on collections of eta quotients with identically vanishing coefficients.

Theorem. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients from one the following lists:*

$$\left\{ f_1^2 C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^3}{f_2^2 f_8} C\left(q^4\right), \frac{f_2^4 f_4}{f_1^2 f_8} C\left(q^4\right) \right\}, \tag{96}$$

$$\left\{ f_1^2 f_2^2 C\left(q^4\right), \frac{f_2^8}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^9}{f_2^4 f_8^3} C\left(q^4\right), \frac{f_2^2 f_4^7}{f_1^2 f_8^3} C\left(q^4\right) \right\}, \tag{97}$$

$$\left\{ \frac{f_3}{f_1} C\left(q^4\right), \frac{f_1 f_4 f_6^3}{f_2^3 f_3 f_{12}} C\left(q^4\right), \frac{f_1 f_2 f_6 f_8^2 f_{12}^2}{f_3 f_4^5 f_{24}} C\left(q^4\right), \frac{f_2^4 f_3 f_8^2 f_{12}^3}{f_1 f_4^6 f_6^2 f_{24}} C\left(q^4\right). \right\}, \tag{98}$$

$$\left\{ \frac{f_1}{f_3} C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6^3} C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_6 f_{24}^2}{f_1 f_8 f_{12}^5} C\left(q^4\right), \frac{f_1 f_4^3 f_6^4 f_{24}^2}{f_2^2 f_3 f_8 f_{12}^6} C\left(q^4\right) \right\}, \tag{99}$$

More New m-Dissection Results, XVI

The 4-dissections above lead to the following theorem on collections of eta quotients with identically vanishing coefficients.

Theorem. *Let* $C(q^4)$ *be any eta quotient with a power series expansion in* q^4*. Let* $F(q)$ *and* $G(q)$ *be any pair of eta quotients from one the following lists:*

$$\left\{ f_1^2 C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^3}{f_2^2 f_8} C\left(q^4\right), \frac{f_2^4 f_4}{f_1^2 f_8} C\left(q^4\right) \right\}, \tag{96}$$

$$\left\{ f_1^2 f_2^2 C\left(q^4\right), \frac{f_2^8}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^9}{f_2^4 f_8^3} C\left(q^4\right), \frac{f_2^2 f_4^7}{f_1^2 f_8^3} C\left(q^4\right) \right\}, \tag{97}$$

$$\left\{ \frac{f_3}{f_1} C\left(q^4\right), \frac{f_1 f_4 f_6^3}{f_2^3 f_3 f_{12}} C\left(q^4\right), \frac{f_1 f_2 f_6 f_8^2 f_{12}^2}{f_3 f_4^5 f_{24}} C\left(q^4\right), \frac{f_2^4 f_3 f_8^2 f_{12}^3}{f_1 f_4^6 f_6^2 f_{24}} C\left(q^4\right). \right\}, \tag{98}$$

$$\left\{ \frac{f_1}{f_3} C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6^3} C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_6 f_{24}^2}{f_1 f_8 f_{12}^5} C\left(q^4\right), \frac{f_1 f_4^3 f_6^4 f_{24}^2}{f_2^2 f_3 f_8 f_{12}^6} C\left(q^4\right) \right\}, \tag{99}$$

More New m-Dissection Results, XVI

The 4-dissections above lead to the following theorem on collections of eta quotients with identically vanishing coefficients.

Theorem. *Let* $C(q^4)$ *be any eta quotient with a power series expansion in* q^4. *Let* $F(q)$ *and* $G(q)$ *be any pair of eta quotients from one the following lists:*

$$\left\{ f_1^2 C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^3}{f_2^2 f_8} C\left(q^4\right), \frac{f_2^4 f_4}{f_1^2 f_8} C\left(q^4\right) \right\}, \tag{96}$$

$$\left\{ f_1^2 f_2^2 C\left(q^4\right), \frac{f_2^8}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^9}{f_2^4 f_8^3} C\left(q^4\right), \frac{f_2^2 f_4^7}{f_1^2 f_8^3} C\left(q^4\right) \right\}, \tag{97}$$

$$\left\{ \frac{f_3}{f_1} C\left(q^4\right), \frac{f_1 f_4 f_6^3}{f_2^3 f_3 f_{12}} C\left(q^4\right), \frac{f_1 f_2 f_6 f_8^2 f_{12}^2}{f_3 f_4^5 f_{24}} C\left(q^4\right), \frac{f_2^4 f_3 f_8^2 f_{12}^3}{f_1 f_4^6 f_6^2 f_{24}} C\left(q^4\right). \right\}, \tag{98}$$

$$\left\{ \frac{f_1}{f_3} C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6^3} C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_6 f_{24}^2}{f_1 f_8 f_{12}^5} C\left(q^4\right), \frac{f_1 f_4^3 f_6^4 f_{24}^2}{f_2^2 f_3 f_8 f_{12}^6} C\left(q^4\right) \right\}, \tag{99}$$

More New m-Dissection Results, XVII

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$$\left\{\frac{f_1^2}{f_6^2}C\left(q^4\right),\frac{f_2^6}{f_1^2f_4^2f_6^2}C\left(q^4\right),\frac{f_1^2f_4^3f_6^2f_{24}^2}{f_2^2f_8f_{12}^6}C\left(q^4\right),\frac{f_2^4f_4f_6^2f_{24}^2}{f_1^2f_8f_{12}^6}C\left(q^4\right)\right\}, \quad (100)$$

$$\left\{\frac{f_3}{f_1f_2^3f_6}C\left(q^4\right),\frac{f_1f_4f_6^2}{f_2^6f_3f_{12}}C\left(q^4\right),\frac{f_2^7f_3f_8^5}{f_1f_4^{15}f_6}C\left(q^4\right),\frac{f_1f_2^4f_6^2f_8^5}{f_3f_4^{14}f_{12}}C\left(q^4\right)\right\}, \quad (101)$$

$$\left\{\frac{f_2^5f_3^2}{f_6}C\left(q^4\right),\frac{f_2^5f_6^5}{f_3^2f_{12}^2}C\left(q^4\right),\frac{f_4^{15}f_6^5}{f_2^5f_3^2f_8^5f_{12}^2}C\left(q^4\right),\frac{f_3^2f_4^{15}}{f_2^5f_6f_8^5}C\left(q^4\right)\right\}, \quad (102)$$

$$\left\{\frac{f_2f_3^2}{f_6}C\left(q^4\right),\frac{f_2f_6^5}{f_3^2f_{12}^2}C\left(q^4\right),\frac{f_4^3f_6^5}{f_2f_3^2f_8f_{12}^2}C\left(q^4\right),\frac{f_3^2f_4^3}{f_2f_6f_8}C\left(q^4\right)\right\}, \quad (103)$$

$$\left\{\frac{f_1f_2^2f_6^2}{f_3}C\left(q^4\right),\frac{f_2^5f_3f_{12}}{f_1f_4f_6}C\left(q^4\right),\frac{f_1f_4^9f_6^2}{f_2^4f_3f_8^3}C\left(q^4\right),\frac{f_3f_4^8f_{12}}{f_1f_2f_6f_8^3}C\left(q^4\right)\right\}, \quad (104)$$

$$\left\{\frac{f_1^2f_6}{f_2^3}C\left(q^4\right),\frac{f_2^3f_6}{f_1^2f_4^2}C\left(q^4\right),\frac{f_1^2f_2f_8^2f_{12}^3}{f_4^6f_6f_{24}}C\left(q^4\right),\frac{f_2^7f_8^2f_{12}^3}{f_1^2f_4^8f_6f_{24}}C\left(q^4\right)\right\}, \quad (105)$$

$$\left\{\frac{f_1^2}{f_6^2}C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2 f_6^2}C\left(q^4\right), \frac{f_1^2 f_4^3 f_6^2 f_{24}^2}{f_2^2 f_8 f_{12}^6}C\left(q^4\right), \frac{f_2^4 f_4 f_6^2 f_{24}^2}{f_1^2 f_8 f_{12}^6}C\left(q^4\right)\right\}, \quad (100)$$

$$\left\{\frac{f_3}{f_1 f_2^3 f_6}C\left(q^4\right), \frac{f_1 f_4 f_6^2}{f_2^6 f_3 f_{12}}C\left(q^4\right), \frac{f_2^7 f_3 f_8^5}{f_1 f_4^{15} f_6}C\left(q^4\right), \frac{f_1 f_2^4 f_6^2 f_8^5}{f_3 f_4^{14} f_{12}}C\left(q^4\right)\right\}, \quad (101)$$

$$\left\{\frac{f_2^5 f_3^2}{f_6}C\left(q^4\right), \frac{f_2^5 f_6^5}{f_3^2 f_{12}^2}C\left(q^4\right), \frac{f_4^{15} f_6^5}{f_2^5 f_3^2 f_8^5 f_{12}^2}C\left(q^4\right), \frac{f_3^2 f_4^{15}}{f_2^5 f_6 f_8^5}C\left(q^4\right)\right\}, \quad (102)$$

$$\left\{\frac{f_2 f_3^2}{f_6}C\left(q^4\right), \frac{f_2 f_6^5}{f_3^2 f_{12}^2}C\left(q^4\right), \frac{f_4^3 f_6^5}{f_2 f_3^2 f_8 f_{12}^2}C\left(q^4\right), \frac{f_3^2 f_4^3}{f_2 f_6 f_8}C\left(q^4\right)\right\}, \quad (103)$$

$$\left\{\frac{f_1 f_2^2 f_6^2}{f_3}C\left(q^4\right), \frac{f_2^5 f_3 f_{12}}{f_1 f_4 f_6}C\left(q^4\right), \frac{f_1 f_4^9 f_6^2}{f_2^4 f_3 f_8^3}C\left(q^4\right), \frac{f_3 f_4^8 f_{12}}{f_1 f_2 f_6 f_8^3}C\left(q^4\right)\right\}, \quad (104)$$

$$\left\{\frac{f_1^2 f_6}{f_2^3}C\left(q^4\right), \frac{f_2^3 f_6}{f_1^2 f_4^2}C\left(q^4\right), \frac{f_1^2 f_2 f_8^2 f_{12}^3}{f_4^6 f_6 f_{24}}C\left(q^4\right), \frac{f_2^7 f_8^2 f_{12}^3}{f_1^2 f_4^8 f_6 f_{24}}C\left(q^4\right)\right\}, \quad (105)$$

$$\left\{\frac{f_1^2}{f_6^2}C\left(q^4\right),\frac{f_2^6}{f_1^2f_4^2f_6^2}C\left(q^4\right),\frac{f_1^2f_4^3f_6^2f_{24}^2}{f_2^2f_8f_{12}^6}C\left(q^4\right),\frac{f_2^4f_4f_6^2f_{24}^2}{f_1^2f_8f_{12}^6}C\left(q^4\right)\right\},\quad(100)$$

$$\left\{\frac{f_3}{f_1f_2^3f_6}C\left(q^4\right),\frac{f_1f_4f_6^2}{f_2^6f_3f_{12}}C\left(q^4\right),\frac{f_2^7f_3f_8^5}{f_1f_4^{15}f_6}C\left(q^4\right),\frac{f_1f_2^4f_6^2f_8^5}{f_3f_4^{14}f_{12}}C\left(q^4\right)\right\},\quad(101)$$

$$\left\{\frac{f_2^5f_3^2}{f_6}C\left(q^4\right),\frac{f_2^5f_6^5}{f_3^2f_{12}^2}C\left(q^4\right),\frac{f_4^{15}f_6^5}{f_2^5f_3^2f_8^5f_{12}^2}C\left(q^4\right),\frac{f_3^2f_4^{15}}{f_2^5f_6f_8^5}C\left(q^4\right)\right\},\quad(102)$$

$$\left\{\frac{f_2f_3^2}{f_6}C\left(q^4\right),\frac{f_2f_6^5}{f_3^2f_{12}^2}C\left(q^4\right),\frac{f_4^3f_6^5}{f_2f_3^2f_8f_{12}^2}C\left(q^4\right),\frac{f_3^2f_4^3}{f_2f_6f_8}C\left(q^4\right)\right\},\quad(103)$$

$$\left\{\frac{f_1f_2^2f_6^2}{f_3}C\left(q^4\right),\frac{f_2^5f_3f_{12}}{f_1f_4f_6}C\left(q^4\right),\frac{f_1f_4^9f_6^2}{f_2^4f_3f_8^3}C\left(q^4\right),\frac{f_3f_4^8f_{12}}{f_1f_2f_6f_8^3}C\left(q^4\right)\right\},\quad(104)$$

$$\left\{\frac{f_1^2f_6}{f_2^3}C\left(q^4\right),\frac{f_2^3f_6}{f_1^2f_4^2}C\left(q^4\right),\frac{f_1^2f_2f_8^2f_{12}^3}{f_4^6f_6f_{24}}C\left(q^4\right),\frac{f_2^7f_8^2f_{12}^3}{f_1^2f_4^8f_6f_{24}}C\left(q^4\right)\right\},\quad(105)$$

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$$\left\{ \frac{f_1^2}{f_6^2} C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2 f_6^2} C\left(q^4\right), \frac{f_1^2 f_4^3 f_6^2 f_{24}^2}{f_2^2 f_8 f_{12}^6} C\left(q^4\right), \frac{f_2^4 f_4 f_6^2 f_{24}^2}{f_1^2 f_8 f_{12}^6} C\left(q^4\right) \right\}, \quad (100)$$

$$\left\{ \frac{f_3}{f_1 f_2^3 f_6} C\left(q^4\right), \frac{f_1 f_4 f_6^2}{f_2^6 f_3 f_{12}} C\left(q^4\right), \frac{f_2^7 f_3 f_8^5}{f_1 f_4^{15} f_6} C\left(q^4\right), \frac{f_1 f_2^4 f_6^2 f_8^5}{f_3 f_4^{14} f_{12}} C\left(q^4\right) \right\}, \quad (101)$$

$$\left\{ \frac{f_2^5 f_3^2}{f_6} C\left(q^4\right), \frac{f_2^5 f_6^5}{f_3^2 f_{12}^2} C\left(q^4\right), \frac{f_4^{15} f_6^5}{f_2^5 f_3^2 f_8^5 f_{12}^2} C\left(q^4\right), \frac{f_3^2 f_4^{15}}{f_2^5 f_6 f_8^5} C\left(q^4\right) \right\}, \quad (102)$$

$$\left\{ \frac{f_2 f_3^2}{f_6} C\left(q^4\right), \frac{f_2 f_6^5}{f_3^2 f_{12}^2} C\left(q^4\right), \frac{f_4^3 f_6^5}{f_2 f_3^2 f_8 f_{12}^2} C\left(q^4\right), \frac{f_3^2 f_4^3}{f_2 f_6 f_8} C\left(q^4\right) \right\}, \quad (103)$$

$$\left\{ \frac{f_1 f_2^2 f_6^2}{f_3} C\left(q^4\right), \frac{f_2^5 f_3 f_{12}}{f_1 f_4 f_6} C\left(q^4\right), \frac{f_1 f_4^9 f_6^2}{f_2^4 f_3 f_8^3} C\left(q^4\right), \frac{f_3 f_4^8 f_{12}}{f_1 f_2 f_6 f_8^3} C\left(q^4\right) \right\}, \quad (104)$$

$$\left\{ \frac{f_1^2 f_6}{f_2^3} C\left(q^4\right), \frac{f_2^3 f_6}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_2 f_8^2 f_{12}^3}{f_4^6 f_6 f_{24}} C\left(q^4\right), \frac{f_2^7 f_8^2 f_{12}^3}{f_1^2 f_4^8 f_6 f_{24}} C\left(q^4\right) \right\}, \quad (105)$$

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$$\left\{ \frac{f_1^2}{f_6^2} C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2 f_6^2} C\left(q^4\right), \frac{f_1^2 f_4^3 f_6^2 f_{24}^2}{f_2^2 f_8 f_{12}^6} C\left(q^4\right), \frac{f_2^4 f_4 f_6^2 f_{24}^2}{f_1^2 f_8 f_{12}^6} C\left(q^4\right) \right\}, \quad (100)$$

$$\left\{ \frac{f_3}{f_1 f_2^3 f_6} C\left(q^4\right), \frac{f_1 f_4 f_6^2}{f_2^6 f_3 f_{12}} C\left(q^4\right), \frac{f_2^7 f_3 f_8^5}{f_1 f_4^{15} f_6} C\left(q^4\right), \frac{f_1 f_2^4 f_6^2 f_8^5}{f_3 f_4^{14} f_{12}} C\left(q^4\right) \right\}, \quad (101)$$

$$\left\{ \frac{f_2^5 f_3^2}{f_6} C\left(q^4\right), \frac{f_2^5 f_6^5}{f_3^2 f_{12}^2} C\left(q^4\right), \frac{f_4^{15} f_6^5}{f_2^5 f_3^2 f_8^5 f_{12}^2} C\left(q^4\right), \frac{f_3^2 f_4^{15}}{f_2^5 f_6 f_8^5} C\left(q^4\right) \right\}, \quad (102)$$

$$\left\{ \frac{f_2 f_3^2}{f_6} C\left(q^4\right), \frac{f_2 f_6^5}{f_3^2 f_{12}^2} C\left(q^4\right), \frac{f_4^3 f_6^5}{f_2 f_3^2 f_8 f_{12}^2} C\left(q^4\right), \frac{f_3^2 f_4^3}{f_2 f_6 f_8} C\left(q^4\right) \right\}, \quad (103)$$

$$\left\{ \frac{f_1 f_2^2 f_6^2}{f_3} C\left(q^4\right), \frac{f_2^5 f_3 f_{12}}{f_1 f_4 f_6} C\left(q^4\right), \frac{f_1 f_4^9 f_6^2}{f_2^4 f_3 f_8^3} C\left(q^4\right), \frac{f_3 f_4^8 f_{12}}{f_1 f_2 f_6 f_8^3} C\left(q^4\right) \right\}, \quad (104)$$

$$\left\{ \frac{f_1^2 f_6}{f_2^3} C\left(q^4\right), \frac{f_2^3 f_6}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_2 f_8^2 f_{12}^3}{f_4^6 f_6 f_{24}} C\left(q^4\right), \frac{f_2^7 f_8^2 f_{12}^3}{f_1^2 f_4^8 f_6 f_{24}} C\left(q^4\right) \right\}, \quad (105)$$

$$\left\{\frac{f_1^2}{f_6^2}C\left(q^4\right), \frac{f_2^6}{f_1^2f_4^2f_6^2}C\left(q^4\right), \frac{f_1^2f_4^3f_6^2f_{24}^2}{f_2^2f_8f_{12}^6}C\left(q^4\right), \frac{f_2^4f_4f_6^2f_{24}^2}{f_1^2f_8f_{12}^6}C\left(q^4\right)\right\}, \quad (100)$$

$$\left\{\frac{f_3}{f_1f_2^3f_6}C\left(q^4\right), \frac{f_1f_4f_6^2}{f_2^6f_3f_{12}}C\left(q^4\right), \frac{f_2^7f_3f_8^5}{f_1f_4^{15}f_6}C\left(q^4\right), \frac{f_1f_2^4f_6^2f_8^5}{f_3f_4^{14}f_{12}}C\left(q^4\right)\right\}, \quad (101)$$

$$\left\{\frac{f_2^5f_3^2}{f_6}C\left(q^4\right), \frac{f_2^5f_6^5}{f_3^2f_{12}^2}C\left(q^4\right), \frac{f_4^{15}f_6^5}{f_2^5f_3^2f_8^5f_{12}^2}C\left(q^4\right), \frac{f_3^2f_4^{15}}{f_2^5f_6f_8^5}C\left(q^4\right)\right\}, \quad (102)$$

$$\left\{\frac{f_2f_3^2}{f_6}C\left(q^4\right), \frac{f_2f_6^5}{f_3^2f_{12}^2}C\left(q^4\right), \frac{f_4^3f_6^5}{f_2f_3^2f_8f_{12}^2}C\left(q^4\right), \frac{f_3^2f_4^3}{f_2f_6f_8}C\left(q^4\right)\right\}, \quad (103)$$

$$\left\{\frac{f_1f_2^2f_6^2}{f_3}C\left(q^4\right), \frac{f_2^5f_3f_{12}}{f_1f_4f_6}C\left(q^4\right), \frac{f_1f_4^9f_6^2}{f_2^4f_3f_8^3}C\left(q^4\right), \frac{f_3f_4^8f_{12}}{f_1f_2f_6f_8^3}C\left(q^4\right)\right\}, \quad (104)$$

$$\left\{\frac{f_1^2f_6}{f_2^3}C\left(q^4\right), \frac{f_2^3f_6}{f_1^2f_4^2}C\left(q^4\right), \frac{f_1^2f_2f_8^2f_{12}^3}{f_4^6f_6f_{24}}C\left(q^4\right), \frac{f_2^7f_8^2f_{12}^3}{f_1^2f_4^8f_6f_{24}}C\left(q^4\right)\right\}, \quad (105)$$

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$$\left\{\frac{f_1 f_6^2}{f_3}C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6}C\left(q^4\right), \frac{f_1 f_4^3 f_6^2}{f_2^2 f_3 f_8}C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_{12}}{f_1 f_6 f_8}C\left(q^4\right)\right\}, \tag{106}$$

$$\left\{\frac{f_1 f_6^7}{f_2 f_3}C\left(q^4\right), \frac{f_2^2 f_3 f_6^4 f_{12}}{f_1 f_4}C\left(q^4\right), \frac{f_2^2 f_3 f_{12}^{16}}{f_1 f_4 f_6^6 f_{24}^5}C\left(q^4\right), \frac{f_1 f_{12}^{15}}{f_2 f_3 f_6^3 f_{24}^5}C\left(q^4\right)\right\}, \tag{107}$$

$$\left\{\frac{f_1^2}{f_2 f_6^5}C\left(q^4\right), \frac{f_2^5}{f_1^2 f_4^2 f_6^5}C\left(q^4\right), \frac{f_1^2 f_6^5 f_{24}^5}{f_2 f_{12}^{15}}C\left(q^4\right), \frac{f_2^5 f_6^5 f_{24}^5}{f_1^2 f_4^2 f_{12}^{15}}C\left(q^4\right)\right\}, \tag{108}$$

$$\left\{\frac{f_1^2}{f_2 f_6}C\left(q^4\right), \frac{f_2^5}{f_1^2 f_4^2 f_6}C\left(q^4\right), \frac{f_1^2 f_6 f_{24}}{f_2 f_{12}^3}C\left(q^4\right), \frac{f_2^5 f_6 f_{24}}{f_1^2 f_4^2 f_{12}^3}C\left(q^4\right)\right\}, \tag{109}$$

$$\left\{\frac{f_1 f_6}{f_2 f_3}C\left(q^4\right), \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6^2}C\left(q^4\right), \frac{f_1 f_6^3 f_{24}}{f_2 f_3 f_{12}^3}C\left(q^4\right), \frac{f_2^2 f_3 f_{24}}{f_1 f_4 f_{12}^2}C\left(q^4\right)\right\}. \tag{110}$$

Then

$$F_{(0)} = G_{(0)}. \tag{111}$$

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$$\left\{ \frac{f_1 f_6^2}{f_3} C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6} C\left(q^4\right), \frac{f_1 f_4^3 f_6^2}{f_2^2 f_3 f_8} C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_{12}}{f_1 f_6 f_8} C\left(q^4\right) \right\}, \tag{106}$$

$$\left\{ \frac{f_1 f_6^7}{f_2 f_3} C\left(q^4\right), \frac{f_2^2 f_3 f_6^4 f_{12}}{f_1 f_4} C\left(q^4\right), \frac{f_2^2 f_3 f_{12}^{16}}{f_1 f_4 f_6^6 f_{24}^5} C\left(q^4\right), \frac{f_1 f_{12}^{15}}{f_2 f_3 f_6^3 f_{24}^5} C\left(q^4\right) \right\}, \tag{107}$$

$$\left\{ \frac{f_1^2}{f_2 f_6^5} C\left(q^4\right), \frac{f_2^5}{f_1^2 f_4^2 f_6^5} C\left(q^4\right), \frac{f_1^2 f_6^5 f_{24}^5}{f_2 f_{12}^{15}} C\left(q^4\right), \frac{f_2^5 f_6^5 f_{24}^5}{f_1^2 f_4^2 f_{12}^{15}} C\left(q^4\right) \right\}, \tag{108}$$

$$\left\{ \frac{f_1^2}{f_2 f_6} C\left(q^4\right), \frac{f_2^5}{f_1^2 f_4^2 f_6} C\left(q^4\right), \frac{f_1^2 f_6 f_{24}}{f_2 f_{12}^3} C\left(q^4\right), \frac{f_2^5 f_6 f_{24}}{f_1^2 f_4^2 f_{12}^3} C\left(q^4\right) \right\}, \tag{109}$$

$$\left\{ \frac{f_1 f_6}{f_2 f_3} C\left(q^4\right), \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6^2} C\left(q^4\right), \frac{f_1 f_6^3 f_{24}}{f_2 f_3 f_{12}^3} C\left(q^4\right), \frac{f_2^2 f_3 f_{24}}{f_1 f_4 f_{12}^2} C\left(q^4\right) \right\}. \tag{110}$$

Then

$$F_{(0)} = G_{(0)}. \tag{111}$$

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$$\left\{\frac{f_1 f_6^2}{f_3}C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6}C\left(q^4\right), \frac{f_1 f_4^3 f_6^2}{f_2^2 f_3 f_8}C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_{12}}{f_1 f_6 f_8}C\left(q^4\right)\right\}, \tag{106}$$

$$\left\{\frac{f_1 f_6^7}{f_2 f_3}C\left(q^4\right), \frac{f_2^2 f_3 f_6^4 f_{12}}{f_1 f_4}C\left(q^4\right), \frac{f_2^2 f_3 f_{12}^{16}}{f_1 f_4 f_6^6 f_{24}^5}C\left(q^4\right), \frac{f_1 f_{12}^{15}}{f_2 f_3 f_6^3 f_{24}^5}C\left(q^4\right)\right\}, \tag{107}$$

$$\left\{\frac{f_1^2}{f_2 f_6^5}C\left(q^4\right), \frac{f_2^5}{f_1^2 f_4^2 f_6^5}C\left(q^4\right), \frac{f_1^2 f_6^5 f_{24}^5}{f_2 f_{12}^{15}}C\left(q^4\right), \frac{f_2^5 f_6^5 f_{24}^5}{f_1^2 f_4^2 f_{12}^{15}}C\left(q^4\right)\right\}, \tag{108}$$

$$\left\{\frac{f_1^2}{f_2 f_6}C\left(q^4\right), \frac{f_2^5}{f_1^2 f_4^2 f_6}C\left(q^4\right), \frac{f_1^2 f_6 f_{24}}{f_2 f_{12}^3}C\left(q^4\right), \frac{f_2^5 f_6 f_{24}}{f_1^2 f_4^2 f_{12}^3}C\left(q^4\right)\right\}, \tag{109}$$

$$\left\{\frac{f_1 f_6}{f_2 f_3}C\left(q^4\right), \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6^2}C\left(q^4\right), \frac{f_1 f_6^3 f_{24}}{f_2 f_3 f_{12}^3}C\left(q^4\right), \frac{f_2^2 f_3 f_{24}}{f_1 f_4 f_{12}^2}C\left(q^4\right)\right\}. \tag{110}$$

Then

$$F_{(0)} = G_{(0)}. \tag{111}$$

More New m-Dissection Results, XVIII

$$\left\{ \frac{f_1 f_6^2}{f_3} C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6} C\left(q^4\right), \frac{f_1 f_4^3 f_6^2}{f_2^2 f_3 f_8} C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_{12}}{f_1 f_6 f_8} C\left(q^4\right) \right\}, \tag{106}$$

$$\left\{ \frac{f_1 f_6^7}{f_2 f_3} C\left(q^4\right), \frac{f_2^2 f_3 f_6^4 f_{12}}{f_1 f_4} C\left(q^4\right), \frac{f_2^2 f_3 f_{12}^{16}}{f_1 f_4 f_6^6 f_{24}^5} C\left(q^4\right), \frac{f_1 f_{12}^{15}}{f_2 f_3 f_6^3 f_{24}^5} C\left(q^4\right) \right\}, \tag{107}$$

$$\left\{ \frac{f_1^2}{f_2 f_6^5} C\left(q^4\right), \frac{f_2^5}{f_1^2 f_4^2 f_6^5} C\left(q^4\right), \frac{f_1^2 f_6^5 f_{24}^5}{f_2 f_{12}^{15}} C\left(q^4\right), \frac{f_2^5 f_6^5 f_{24}^5}{f_1^2 f_4^2 f_{12}^{15}} C\left(q^4\right) \right\}, \tag{108}$$

$$\left\{ \frac{f_1^2}{f_2 f_6} C\left(q^4\right), \frac{f_2^5}{f_1^2 f_4^2 f_6} C\left(q^4\right), \frac{f_1^2 f_6 f_{24}}{f_2 f_{12}^3} C\left(q^4\right), \frac{f_2^5 f_6 f_{24}}{f_1^2 f_4^2 f_{12}^3} C\left(q^4\right) \right\}, \tag{109}$$

$$\left\{ \frac{f_1 f_6}{f_2 f_3} C\left(q^4\right), \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6^2} C\left(q^4\right), \frac{f_1 f_6^3 f_{24}}{f_2 f_3 f_{12}^3} C\left(q^4\right), \frac{f_2^2 f_3 f_{24}}{f_1 f_4 f_{12}^2} C\left(q^4\right) \right\}. \tag{110}$$

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Then

$$F_{(0)} = G_{(0)}. \tag{111}$$

More New m-Dissection Results, XIX

Now $C(q^4)$ can be specialized in any of the collections above to prove vanishing coefficient results for collections of eta quotients which experiment indicated had vanishing coefficient similar to that of one of f_1^r, $r = 4, 6, 8, 10, 14$ and 26 or $f_1^3 f_2^3$.

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Together, these allow some of the "fine structure" of the tables/graphs to be proven.

We close with two examples.

A Collection of Eta Quotients with Identically Vanishing Coefficients

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Let $F(q)$ and $G(q)$ be any two eta quotients from the following collection (which is from the table/graph for f_1^4):

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Then Then

$$F_{(0)} = G_{(0)}.$$

An Example of Strict Inclusion of Sets of Vanishing Coefficients

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The following pair of collections of eta quotients are also from the table/graph for f_1^4 (actually VIII is the collection in the previous example) :

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$$VIII = \left\{ \frac{f_2^3 f_3 f_8 f_{12}^8}{f_1 f_4^3 f_6^4 f_{24}^3}, \frac{f_1 f_8 f_{12}^7}{f_3 f_4^2 f_6 f_{24}^3}, \frac{f_1 f_4^8 f_6^3 f_{24}}{f_2^4 f_3 f_8^3 f_{12}^3}, \frac{f_3 f_4^7 f_{24}}{f_1 f_2 f_8^3 f_{12}^2}, \frac{f_1 f_2^2 f_6}{f_3 f_4}, \frac{f_2^5 f_3 f_{12}}{f_1 f_4^2 f_6^2}, \frac{f_1 f_4 f_6^5}{f_2^2 f_3 f_{12}^2}, \frac{f_2 f_3 f_6^2}{f_1 f_{12}} \right\},$$

$$XIV = \left\{ \frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}, \frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}} \right\}.$$

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$$XIV = \left\{ \frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}, \frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}} \right\}.$$

If $A(q)$ is any of the 8 eta quotients in collection VIII and $B(q)$ is either of the 2 eta quotients in collection XIV,

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$$XIV = \left\{ \frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}, \frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}} \right\}.$$

If $A(q)$ is any of the 8 eta quotients in collection VIII and $B(q)$ is either of the 2 eta quotients in collection XIV, then

$$A_{(0)} \subsetneq B_{(0)}.$$

The Table for f_1^4

Table 9: Eta quotients with vanishing behaviour similar to f_1^4

Collection	# of eta quotients	Collection	# of eta quotients
I	72	II *	4
III †	2	IV	6
V †	2	VI *	4
VII *	6	VIII *	8
IX *	4	X	4
XI	14	XII †	2
XIII †	2	XIV †	2
XV	4	XVI †	2
XVII	4	XVIII †	2
XIX †	6		

The Graph for f_1^4

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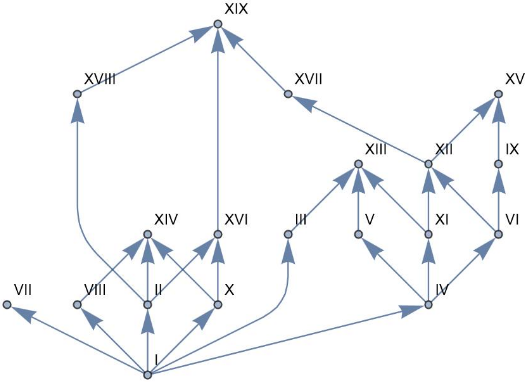

Figure: The grouping of the 150 eta-quotients in Table 9, which have vanishing coefficient behaviour similar to f_1^4

Thanks

Thank you for listening/watching.