SOME IMPLICATIONS OF THE WP-BAILEY TREE

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Abstract. We consider a special case of a WP-Bailey chain of George Andrews, and use it to derive a number of curious transformations of basic hypergeometric series.

We also derive two new WP-Bailey pairs, and use them to derive some new transformations for basic hypergeometric series.

Finally, we briefly consider the implications of WP-Bailey pairs $(\alpha_n(a,k),\, \beta_n(a,k))$, in which $\alpha_n(a,k)$ is independent of k, for generalizations of identities of the Rogers-Ramanujan type.

1. Introduction

Andrews, building on prior work of Bressoud [5] and Singh [9], in [1] defined a *WP-Bailey pair* to be a pair of sequences $(\alpha_n(a,k),\, \beta_n(a,k))$ satisfying

$$\begin{aligned}\beta_n(a,k) &= \sum_{j=0}^{n} \frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k) \\ &= \frac{(k/a,k;q)_n}{(aq,q;q)_n}\sum_{j=0}^{n}\frac{(q^{-n})_j(kq^n)_j}{(aq^{1-n}/k)_j(aq^{n+1})_j}\left(\frac{qa}{k}\right)^j \alpha_j(a,k).\end{aligned} \tag{1.1}$$

Andrews also showed in [1] that there were two distinct ways to construct new WP-Bailey pairs from a given pair. If $(\alpha_n(a,k),\, \beta_n(a,k))$ satisfy (1.1), then so do $(\alpha'_n(a,k),\, \beta'_n(a,k))$ and $(\tilde{\alpha}_n(a,k),\, \tilde{\beta}_n(a,k))$, where

$$\begin{aligned}\alpha'_n(a,k) &= \frac{(\rho_1,\rho_2)_n}{(aq/\rho_1,aq/\rho_2)_n}\left(\frac{k}{c}\right)^n \alpha_n(a,c), \\ \beta'_n(a,k) &= \frac{(k\rho_1/a,k\rho_2/a)_n}{(aq/\rho_1,aq/\rho_2)_n} \\ &\quad\times \sum_{j=0}^{n}\frac{(1-cq^{2j})(\rho_1,\rho_2)_j(k/c)_{n-j}(k)_{n+j}}{(1-c)(k\rho_1/a,k\rho_2/a)_n(q)_{n-j}(qc)_{n+j}}\left(\frac{k}{c}\right)^j\beta_j(a,c),\end{aligned} \tag{1.2}$$

Date: November 14, 2008.

2000 *Mathematics Subject Classification.* Primary: 33D15. Secondary:11B65, 05A19.

Key words and phrases. Q-Series, Rogers-Ramanujan Type Identities, Bailey chains, finite Rogers-Ramanujan identities, WP-Bailey pairs.

with $c = k\rho_1\rho_2/aq$ for the pair above, and

$$\tilde{\alpha}_n(a,k) = \frac{(qa^2/k)_{2n}}{(k)_{2n}}\left(\frac{k^2}{qa^2}\right)^n \alpha_n\left(a, \frac{qa^2}{k}\right), \tag{1.3}$$

$$\tilde{\beta}_n(a,k) = \sum_{j=0}^{n} \frac{(k^2/qa^2)_{n-j}}{(q)_{n-j}}\left(\frac{k^2}{qa^2}\right)^j \beta_j\left(a, \frac{qa^2}{k}\right).$$

These two constructions allow a "tree" of WP-Bailey pairs to be generated from a single WP-Bailey pair. Andrews and Berkovich [2] further investigated these two branches of the WP-Bailey tree, in the process deriving many new transformations for basic hypergeometric series. Spiridonov [11] derived an elliptic generalization of Andrews first WP-Bailey chain, and Warnaar [14] [1] added four new branches to the WP-Bailey tree, two of which had generalizations to the elliptic level. More recently, Liu and Ma [7] introduced the idea of a general WP-Bailey chain (as a solution to a system of linear equations), and added one new branch to the WP-Bailey tree.

In the present paper, we derive two new WP-Bailey pairs, one of them restricted in the sense that it is necessary to set $k = q$. We then insert these in some of the WP-Bailey chains listed above to derive new transformations of basic hypergeometric series.

We also consider a special case ($k = aq$) of the first WP-Bailey chain of Andrews, and show how it leads some unusual transformations of series.

We also briefly consider the special case of a WP-Bailey pair ($\alpha_n(a)$, $\beta_n(a,k)$), where the α_n are independent of k. We show how a such pair may give rise to a generalization of a Slater-type identity deriving from the standard Bailey pair $(\alpha_n(a), \beta_n(a,0))$.

2. Finite Basic Hypergeometric Identities deriving from "trivial" WP-Bailey pairs

In this section we derive some unusual transformations from (1.2) by inserting "trivial" WP-Bailey pairs (see below for the definition).

We reformulate the constructions at (1.2) and (for later use) (1.3) as transformations relating the original WP-Bailey pair ($\alpha_n(a,k)$, $\beta_n(a,k)$). We first recall the following elementary transformation:

$$(a;q)_{n-r} = \frac{q^{r(r+1)/2}(a;q)_n}{(q^{1-n}/a;q)_r(-aq^n)^r}. \tag{2.1}$$

Theorem 1. *If* $(\alpha_n(a,k), \beta_n(a,k))$ *satisfy*

$$\beta_n(a,k) = \sum_{j=0}^{n} \frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k), \tag{2.2}$$

[1] In a note added after submitting the paper [14], Warnaar remarks that he had discovered many more transformations for basic and elliptic WP-Bailey pairs, and gives two further examples of chains that hold at the elliptic level.

then

$$(2.3)\quad \sum_{n=0}^{N} \frac{(1-kq^{2n})(\rho_1,\rho_2,kaq^{N+1}/\rho_1\rho_2,q^{-N};q)_n}{(1-k)(kq/\rho_1,kq/\rho_2,\rho_1\rho_2q^{-N}/a,kq^{1+N};q)_n} q^n\beta_n(a,k)$$
$$= \frac{(kq,kq/\rho_1\rho_2,aq/\rho_1,aq/\rho_2;q)_N}{(kq/\rho_1,kq/\rho_2,aq/\rho_1\rho_2,aq;q)_N}$$
$$\times \sum_{n=0}^{N} \frac{(\rho_1,\rho_2,kaq^{N+1}/\rho_1\rho_2,q^{-N};q)_n}{(aq/\rho_1,aq/\rho_2,aq^{1+N},\rho_1\rho_2q^{-N}/k;q)_n} \left(\frac{aq}{k}\right)^n \alpha_n(a,k),$$

and

$$(2.4)\quad \sum_{n=0}^{N} \frac{(q^{-N};q)_n}{(q^{-N}k^2/a^2;q)_n} q^n\beta_n(a,k) = \frac{(qa/k,qa^2/k;q)_N}{(qa,qa^2/k^2;q)_N}$$
$$\times \sum_{n=0}^{N} \frac{(q^{N+1}a^2/k,q^{-N};q)_n(k;q)_{2n}}{(aq^{1+N},q^{-N}k/a;q)_n(qa^2/k;q)_{2n}} \left(\frac{aq}{k}\right)^n \alpha_n(a,k).$$

Proof. The identity at (2.3) follows from (1.2), after substituting for $\alpha'_n(a,k)$ in (1.1), employing (2.1), then setting the two expressions for $\beta'_n(a,k)$ equal, and finally replacing c with k. The identity at (2.4) follows from (1.3), after setting $k = qa^2/c$, using (2.1), and finally replacing c with k. □

For later use we also note the following corollary, which is immediate upon letting $N \to \infty$.

Corollary 1. *If* $(\alpha_n(a,k),\beta_n(a,k))$ *satisfy*

$$\beta_n(a,k) = \sum_{j=0}^{n} \frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k),$$

then

$$(2.5)\quad \sum_{n=0}^{\infty} \frac{(1-kq^{2n})(\rho_1,\rho_2;q)_n}{(1-k)(kq/\rho_1,kq/\rho_2;q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n(a,k) =$$
$$\frac{(kq,kq/\rho_1\rho_2,aq/\rho_1,aq/\rho_2;q)_\infty}{(kq/\rho_1,kq/\rho_2,aq/\rho_1\rho_2,aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a,k),$$

and

$$(2.6)\quad \sum_{n=0}^{\infty} \left(\frac{qa^2}{k^2}\right)^n \beta_n(a,k)$$
$$= \frac{(qa/k,qa^2/k;q)_\infty}{(qa,qa^2/k^2;q)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(qa^2/k;q)_{2n}} \left(\frac{qa^2}{k^2}\right)^n \alpha_n(a,k).$$

Remark: At several places throughout the paper we replace a with x/q, ρ_1 with y and ρ_2 with z, in order to more easily make comparisons with a transformation due to Bailey (4.4) later.

We now consider some applications of the following corollary.

Corollary 2. *Let N be a positive integer. Suppose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are related by*

$$\beta_n = \sum_{r=0}^{n} \alpha_r. \tag{2.7}$$

Then

$$\begin{aligned}\sum_{n=0}^{N} \frac{(q\sqrt{k}, -q\sqrt{k}, y, z, k^2q^N/yz, q^{-N}; q)_n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, yzq^{1-N}/k, kq^{1+N}; q)_n} q^n \beta_n \\ = \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \\ \times \sum_{n=0}^{N} \frac{(y, z, k^2q^N/yz, q^{-N}; q)_n}{(k/y, k/z, kq^N, yzq^{-N}/k; q)_n} \alpha_n.\end{aligned} \tag{2.8}$$

Proof. Let $\rho_1 = y$, $\rho_2 = z$ and $a = k/q$ in (2.2) and (2.3) in Theorem 1. □

We could reasonably call pairs (α_n, β_n) satisfying (2.7) *trivial WP-Bailey pairs*. It is interesting that the simple condition relating α_n and β_n at (2.7) should have several non-trivial consequences. Before noticing the result in Corollary 2, we had first remarked in [8] the result that derives from it by letting $N \to \infty$ (found by another method), so in fact many of the results in [8] has a finite counterpart. We give some examples.

Corollary 3. *Let N be a positive integer. Then*

$$\begin{aligned}\sum_{n=0}^{\lfloor N/2 \rfloor} \frac{(q\sqrt{k}, -q\sqrt{k}, y, z, k^2q^N/yz, q^{-N}; q)_{2n} q^{2n}}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, yzq^{1-N}/k, kq^{1+N}; q)_{2n}} \\ = \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \\ \times \sum_{n=0}^{N} \frac{(y, z, k^2q^N/yz, q^{-N}; q)_n}{(k/y, k/z, kq^N, yzq^{-N}/k; q)_n} (-1)^n.\end{aligned} \tag{2.9}$$

Proof. Let $\alpha_n = (-1)^n$ in Corollary 2. □

Remark: The left side in (2.9) above could be expressed (see [6, page 39] for an explanation of the notation) as

$${}_{12}W_{11}\left(k; y, yq, z, zq, \frac{k^2q^N}{yz}, \frac{k^2q^{1+N}}{yz}, q^{-N}, q^{1-N}, q^2; q^2; q^2\right),$$

where the additional q-products are inserted here to give the series the form of a ${}_{r+1}W_r$ series.

Corollary 4. *Let N be a positive integer. Then*

$$\sum_{n=0}^{N} \frac{(q\sqrt{k}, -q\sqrt{k}, y, z, k^2q^N/yz, q^{-N}; q)_n (n+1)\, q^n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, yzq^{1-N}/k, kq^{1+N}; q)_n}$$
$$= \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \times \sum_{n=0}^{N} \frac{(y, z, k^2q^N/yz, q^{-N}; q)_n}{(k/y, k/z, kq^N, yzq^{-N}/k; q)_n}.$$

Proof. Let $\alpha_n = 1$ in Corollary 2. □

Corollary 5.

$$(2.10)\quad {}_8\phi_7\left[\begin{matrix} q\sqrt{k}, -q\sqrt{k}, y, z, aq, bq, k^2q^N/yz, q^{-N} \\ \sqrt{k}, -\sqrt{k}, qk/y, qk/z, (a+b-1)q, yzq^{1-N}/k, kq^{1+N} \end{matrix}; q, q\right]$$
$$= \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \times {}_6\phi_5\left[\begin{matrix} y, z, a, b, k^2q^N/yz, q^{-N} \\ k/y, k/z, (a+b-1)q, kq^N, yzq^{-N}/k \end{matrix}; q, q^2\right].$$

Proof. In Corollary 2, define $\alpha_0 = 1$, and for $n > 0$,

$$\alpha_n = \frac{(aq, bq; q)_n}{((a+b-1)q, q; q)_n} - \frac{(aq, bq; q)_{n-1}}{((a+b-1)q, q; q)_{n-1}} = \frac{(a, b; q)_n q^{2n}}{((a+b-1)q, q; q)_n}.$$

The sum for β_n is easily seen to telescope to give

$$\beta_n = \frac{(aq, bq; q)_n}{((a+b-1)q, q; q)_n}.$$

and the result follows. □

Corollary 6.

$$(2.11)\quad {}_8\phi_7\left[\begin{matrix} q\sqrt{k}, -q\sqrt{k}, y, z, aq, bq, k^2q^N/yz, q^{-N} \\ \sqrt{k}, -\sqrt{k}, qk/y, qk/z, abq, yzq^{1-N}/k, kq^{1+N} \end{matrix}; q, q\right]$$
$$= \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \times {}_6\phi_5\left[\begin{matrix} y, z, a, b, k^2q^N/yz, q^{-N} \\ k/y, k/z, abq, kq^N, yzq^{-N}/k \end{matrix}; q, q\right].$$

Proof. This time, in Corollary 2, define $\alpha_0 = 1$, and for $n > 0$,

$$\alpha_n = \frac{(aq, bq; q)_n}{(abq, q; q)_n} - \frac{(aq, bq; q)_{n-1}}{(abq, q; q)_{n-1}} = \frac{(a, b; q)_n q^n}{(abq, q; q)_n}.$$

The result follows as above. □

Corollary 7. *Let N and m be positive integers and let p be an integer. Then*

$$\sum_{n=0}^{N} \frac{(q\sqrt{k}, -q\sqrt{k}, y, z, k^2q^N/yz, q^{-N}; q)_n q^{(mn^2+(p+2)n)/2}}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, yzq^{1-N}/k, kq^{1+N}; q)_n (-q^{(m+p)/2}; q^m)_n}$$
$$= \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)}$$
$$\times \left(1 - q^{(m-p)/2} \sum_{n=1}^{N} \frac{(y, z, k^2q^N/yz, q^{-N}; q)_n q^{(mn^2+(p-2m)n)/2}}{(k/y, k/z, kq^N, yzq^{-N}/k; q)_n (-q^{(m+p)/2}; q^m)_n}\right).$$

Proof. In Corollary 2 set $\alpha_0 = 1$ and, for $n > 0$,

$$\alpha_n = \frac{q^{(mn^2+pn)/2}}{(-q^{(m+p)/2}; q^m)_n} - \frac{q^{(m(n-1)^2+p(n-1))/2}}{(-q^{(m+p)/2}; q^m)_{n-1}} = -q^{(m-p)/2} \frac{q^{(mn^2+(p-2m)n)/2}}{(-q^{(m+p)/2}; q^m)_n}.$$

□

Corollary 8. *Let P, p, Q, q, R, a, b, c, k, y and z be complex numbers such that none of the denominators below vanish. Then*

$$\sum_{n=0}^{N} \frac{\left(q\sqrt{k}, -q\sqrt{k}, y, z, \frac{k^2q^N}{yz}, q^{-N}; q\right)_n}{\left(\sqrt{k}, -\sqrt{k}, \frac{qk}{y}, \frac{qk}{z}, \frac{yzq^{1-N}}{k}, kq^{1+N}; q\right)_n} \tag{2.12}$$
$$\times \frac{(ap^2; p^2)_n \left(bP^2; P^2\right)_n (cR^2; R^2)_n \left(\frac{aQ^2}{bc}; Q^2\right)_n q^n}{\left(\frac{PQR}{p}; \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}; \frac{pPQ}{R}\right)_n \left(\frac{apQR}{bP}; \frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q}; \frac{pPR}{Q}\right)_n}$$
$$= \frac{\left(1-\frac{k}{y}\right)\left(1-\frac{k}{z}\right)(1-kq^N)\left(1-\frac{kq^N}{yz}\right)}{(1-k)\left(1-\frac{k}{yz}\right)\left(1-\frac{kq^N}{y}\right)\left(1-\frac{kq^N}{z}\right)} \sum_{n=0}^{N} \frac{\left(y, z, \frac{k^2q^N}{yz}, q^{-N}; q\right)_n}{\left(\frac{k}{y}, \frac{k}{z}, kq^N, \frac{yzq^{-N}}{k}; q\right)_n}$$
$$\times \frac{(1-ap^nP^nQ^nR^n)\left(1-b\frac{p^nP^n}{Q^nR^n}\right)\left(1-\frac{P^nQ^n}{cp^nR^n}\right)\left(1-\frac{ap^nQ^n}{bcP^nR^n}\right)}{(1-a)(1-b)\left(1-\frac{1}{c}\right)\left(1-\frac{a}{bc}\right)}$$
$$\times \frac{(a; p^2)_n \left(b; P^2\right)_n (c; R^2)_n \left(\frac{a}{bc}; Q^2\right)_n R^{2n}}{\left(\frac{PQR}{p}; \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}; \frac{pPQ}{R}\right)_n \left(\frac{apQR}{bP}; \frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q}; \frac{pPR}{Q}\right)_n};$$

(2.13)
$$\sum_{n=0}^{N} \frac{\left(q\sqrt{k}, -q\sqrt{k}, y, z, \frac{k^2q^N}{yz}, q^{-N}; q\right)_n}{\left(\sqrt{k}, -\sqrt{k}, \frac{qk}{y}, \frac{qk}{z}, \frac{yzq^{1-N}}{k}, kq^{1+N}; q\right)_n} \frac{(aq^m, bq^m, cq^m, \frac{aq^m}{bc}; q^m)_n}{\left(\frac{a}{c}q^m, \frac{a}{b}q^m, bcq^m, q^m; q^m\right)_n} q^n$$
$$= \frac{\left(1-\frac{k}{y}\right)\left(1-\frac{k}{z}\right)(1-kq^N)\left(1-\frac{kq^N}{yz}\right)}{(1-k)\left(1-\frac{k}{yz}\right)\left(1-\frac{kq^N}{y}\right)\left(1-\frac{kq^N}{z}\right)} \times$$

$$\sum_{n=0}^{N} \frac{\left(y, z, \frac{k^2q^N}{yz}, q^{-N}; q\right)_n}{\left(\frac{k}{y}, \frac{k}{z}, kq^N, \frac{yzq^{-N}}{k}; q\right)_n} \frac{(q^m\sqrt{a}, -q^m\sqrt{a}, a, b, c, \frac{a}{bc}; q^m)_n \, q^{mn}}{\left(\sqrt{a}, -\sqrt{a}, \frac{a}{c}q^m, \frac{a}{b}q^m, bcq^m, q^m; q^m\right)_n}.$$

Proof. We use the special case $m = 0$, $d = 1$ of the identity of Subbarao and Verma labeled (2.2) in [12], namely,

$$(2.14)\quad \sum_{k=0}^{n} \frac{(1 - ap^kP^kQ^kR^k)\left(1 - b\frac{p^kP^k}{Q^kR^k}\right)\left(1 - \frac{P^kQ^k}{cp^kR^k}\right)\left(1 - \frac{ap^kQ^k}{bcP^kR^k}\right)}{(1-a)(1-b)\left(1-\frac{1}{c}\right)\left(1-\frac{a}{bc}\right)}$$
$$\times \frac{(a;p^2)_k\left(b;P^2\right)_k}{\left(\frac{PQR}{p};\frac{PQR}{p}\right)_k\left(\frac{apPQ}{cR};\frac{pPQ}{R}\right)_k} \frac{(c;R^2)_k\left(\frac{a}{bc};Q^2\right)_k}{\left(\frac{apQR}{bP};\frac{pQR}{P}\right)_k\left(\frac{bcpPR}{Q};\frac{pPR}{Q}\right)_k} R^{2k}$$
$$= \frac{(ap^2;p^2)_n\left(bP^2;P^2\right)_n(cR^2;R^2)_n\left(\frac{aQ^2}{bc};Q^2\right)_n}{\left(\frac{PQR}{p};\frac{PQR}{p}\right)_n\left(\frac{apPQ}{cR};\frac{pPQ}{R}\right)_n\left(\frac{apQR}{bP};\frac{pQR}{P}\right)_n\left(\frac{bcpPR}{Q};\frac{pPR}{Q}\right)_n},$$

and then, in (2.8) above, let α_i be the i-th term in the sum above, and let β_n be the quantity on the right side above.

The identity at (2.13) follows upon setting $P = Q = p = R = q^{m/2}$ and simplifying. □

Apart from the first chain of Andrews, there is only one other WP-Bailey chain, of those seven alluded to in the introduction, that leads to a non-trivial transformation similar to that in Corollary 2 (the consequences of letting $k = aq$ in the other chains being entirely trivial). This is the third chain of Warnaar (Theorem 2.5 in [14]). This chain implies that if $(\alpha_n(a,k), \beta_n(a,k))$ satisfy (1.1), then

$$(2.15)\quad \sum_{j=0}^{n} \frac{1+aq^{2j}}{1+a} \frac{(m/a;q^2)_{n-j}(am;q^2)_{n+j}}{(q^2;q^2)_{n-j}(a^2q^2;q^2)_{n+j}} q^{-j}\alpha_j(a,m)$$
$$= q^{-n}\frac{(-mq;q)_{2n}}{(-a;q)_{2n}}$$
$$\times \sum_{j=0}^{n} \frac{1-mq^{2j}}{1-m} \frac{(a/m;q^2)_{n-j}(am;q^2)_{n+j}}{(q^2;q^2)_{n-j}(m^2q^2;q^2)_{n+j}} \left(\frac{m}{a}\right)^{n-j} \beta_j(a,m).$$

Upon setting $m = aq$, we get that if $\beta_n = \sum_{j=0}^{n} \alpha_j$, then

$$(2.16)\quad \sum_{j=0}^{n} \frac{1+aq^{2j}}{1+a} \frac{(q;q^2)_{n-j}(a^2q;q^2)_{n+j}}{(q^2;q^2)_{n-j}(a^2q^2;q^2)_{n+j}} q^{-j}\alpha_j$$
$$= \frac{(1+aq^{2n})(1+aq^{2n+1})}{(1+a)(1+aq)} \sum_{j=0}^{n} \frac{1-aq^{2j+1}}{1-aq} \frac{(1/q;q^2)_{n-j}(a^2q;q^2)_{n+j}}{(q^2;q^2)_{n-j}(a^2q^4;q^2)_{n+j}} q^{-j}\beta_j.$$

However, we do not pursue the consequences of this transformation further here.

3. New WP-Bailey Pairs

We next exhibit two new WP-Bailey pairs.

Lemma 1. *The pair* $(\alpha_n^{(1)}(a,k), \beta_n^{(1)}(a,k))$ *is a WP-Bailey pair, where*

$$\alpha_n^{(1)}(a,k) = \frac{(qa^2/k^2;q)_n}{(q;q)_n}\left(\frac{k}{a}\right)^n, \tag{3.1}$$
$$\beta_n^{(1)}(a,k) = \frac{(qa/k,k;q)_n}{(k^2/a,q;q)_n}\frac{(k^2/a;q)_{2n}}{(aq;q)_{2n}}.$$

Proof.

$$\sum_{j=0}^{n}\frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j^{(1)}(a,k)$$
$$= \frac{(k/a,k;q)_n}{(aq,q;q)_n}\sum_{j=0}^{n}\frac{(q^{-n},kq^n;q)_j}{(aq^{1-n}/k,aq^{n+1};q)_j}\left(\frac{qa}{k}\right)^j\alpha_j^{(1)}(a,k)$$
$$= \frac{(k/a,k;q)_n}{(aq,q;q)_n}\sum_{j=0}^{n}\frac{(q^{-n},kq^n,qa^2/k^2;q)_j}{(aq^{1-n}/k,aq^{n+1},q;q)_j}q^j$$
$$= \frac{(k/a,k;q)_n}{(aq,q;q)_n}\frac{(aq/k,k^2q^n/a;q)_n}{(aq^{n+1},k/a;q)_n}$$
$$= \frac{(qa/k,k;q)_n}{(k^2/a,q,q)_n}\frac{(k^2/a;q)_{2n}}{(aq,q)_{2n}} = \beta_n^{(1)}(a,k).$$

The third equality follows from the q-Pfaff-Saalschütz sum,

$${}_3\phi_2\left[\begin{matrix}a,b,q^{-n}\\ c,abq^{1-n}/c\end{matrix};q,q\right] = \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}. \tag{3.2}$$

□

Lemma 2. *The pair* $(\alpha_n^{(2)}(a,q), \beta_n^{(2)}(a,q))$ *is a WP-Bailey pair, where*

$$\alpha_n^{(2)}(a,q) = \frac{(a,\,q\sqrt{a},\,-q\sqrt{a},\,d,\,q/d,\,-a;q)_n}{(\sqrt{a},\,-\sqrt{a},\,aq/d,\,ad,\,-q,\,q;q)_n}(-1)^n, \tag{3.3}$$
$$\beta_n^{(2)}(a,q) = \begin{cases}\dfrac{(q^2/ad,dq/a;q^2)_{n/2}}{(adq,aq^2/d;q^2)_{n/2}}, & n \text{ even},\\ -a\dfrac{(q/ad,d/a;q^2)_{(n+1)/2}}{(ad,aq/d;q^2)_{(n+1)/2}}, & n \text{ odd}.\end{cases}$$

Remark: Note that this pair is restricted in that it is necessary to set $k = q$ for (1.1) to hold.

Proof.

$$\begin{aligned}
&\sum_{j=0}^{n} \frac{(q/a)_{n-j}(q)_{n+j}}{(q)_{n-j}(aq)_{n+j}} \alpha_j^{(2)}(a,q)\\
&= \frac{(q/a;q)_n}{(aq;q)_n} \sum_{j=0}^{n} \frac{(q^{-n},q^{n+1};q)_j}{(aq^{-n},aq^{n+1};q)_j} a^j \alpha_j^{(2)}(a,q)\\
&= \frac{(q/a;q)_n}{(aq;q)_n} \sum_{j=0}^{n} \frac{(q^{-n},q^{n+1},a,q\sqrt{a},-q\sqrt{a},d,\frac{q}{d},-a;q)_j}{(aq^{-n},aq^{n+1},\sqrt{a},-\sqrt{a},\frac{qa}{d},ad,-q,q;q)_j}(-a)^j\\
&= \frac{(q/a;q)_n}{(aq;q)_n} \begin{cases} -a\dfrac{(aq;q)_n}{(q/a;q)_n}\dfrac{(q/ad,d/a;q^2)_{(n+1)/2}}{(ad,qa/d;q^2)_{(n+1)/2}}, & n \text{ odd}\\ \dfrac{(aq;q)_n}{(q/a;q)_n}\dfrac{(q^2/ad,dq/a;q^2)_{n/2}}{(adq,q^2a/d;q^2)_{n/2}}, & n \text{ even,} \end{cases}\\
&= \beta_n^{(2)}(a,q).
\end{aligned}$$

The third equality follows from a q-analogue of Whipple's ${}_3F_2$ sum,

$$(3.4)\qquad {}_8\phi_7\left[\begin{matrix} C,\ q\sqrt{C},\ -q\sqrt{C},\ a,\ q/a,\ -C,\ d,\ q/d \\ \sqrt{C},\ -\sqrt{C},\ Cq/a,\ aC,\ -q,\ Cq/d,\ Cd \end{matrix}; q, -C\right] = \frac{(C,Cq;q)_\infty(aCd,aCq/d,Cdq/a,Cq^2/ad;q^2)_\infty}{(Cd,Cq/d,aC,Cq/a;q)_\infty},$$

and upon setting $a = q^{-n}$,

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} C,\ q\sqrt{C},\ -q\sqrt{C},\ d,\ q/d,\ -C,\ q^{-n},\ q^{1+n} \\ \sqrt{C},\ -\sqrt{C},\ Cq/d,\ dC,\ -q,\ Cq^{1+n},\ Cq^{-n} \end{matrix}; q, -C\right]\\
&\qquad= \begin{cases} \dfrac{(Cq;q)_n}{(q/C;q)_n}\dfrac{(q^2/dC,dq/C;q^2)_{n/2}}{(dCq,Cq^2/d;q^2)_{n/2}}, & n \text{ even}\\ \dfrac{(Cq;q)_n}{(q/C;q)_n}\dfrac{(q/dC,d/C;q^2)_{(n+1)/2}}{(dC,Cq/d;q^2)_{(n+1)/2}}(-C), & n \text{ odd.} \end{cases}
\end{aligned}$$

□

We had initially thought that the following WP-Bailey pair was totally new also.

$$(3.5)\qquad \begin{aligned}
\alpha_n^{(3)}(a,k) &= \frac{(a,\ q\sqrt{a},\ -q\sqrt{a},\ k/a,\ a\sqrt{q/k},\ -a\sqrt{q/k};q)_n}{(\sqrt{a},\ -\sqrt{a},\ qa^2/k,\ \sqrt{qk},\ -\sqrt{qk},\ q;q)_n}(-1)^n,\\
\beta_n^{(3)}(a,k) &= \begin{cases} \dfrac{(k,k^2/a^2;q^2)_{n/2}}{(q^2,q^2a^2/k;q^2)_{n/2}}, & n \text{ even,}\\ 0, & n \text{ odd.} \end{cases}
\end{aligned}$$

However, this pair can be derived from an existing WP-Bailey pair using a result of Warnaar [14].

Lemma 3 (Warnaar, [14]). *For a and k indeterminates the following equations are equivalent:*

(3.6)
$$\beta_n(a,k;q) = \sum_{j=0}^{n} \frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}} \alpha_j(a,k;q),$$
$$\alpha_n(a,k;q) = \frac{1-aq^{2n}}{1-a} \sum_{j=0}^{n} \frac{1-kq^{2n}}{1-k} \frac{(a/k)_{n-j}(a)_{n+j}}{(q)_{n-j}(kq)_{n+j}} \left(\frac{k}{a}\right)^{n-j} \beta_j(a,k;q).$$

If a and k are interchanged in the second equation, we see that Lemma 3 implies the following.

Corollary 9. *If $(\alpha_n(a,k), \beta_n(a,k))$ are a WP-Bailey pair, then so are $(\alpha'_n(a,k), \beta'_n(a,k))$, where*
$$\alpha'_n(a,k) = \frac{1-aq^{2n}}{1-a} \left(\frac{k}{a}\right)^n \beta_n(k,a),$$
$$\beta'_n(a,k) = \frac{1-k}{1-kq^{2n}} \left(\frac{k}{a}\right)^n \alpha_n(k,a).$$

For the present purposes, we may call the pair $(\alpha'_n(a,k), \beta'_n(a,k))$ the *dual* of the pair $(\alpha_n(a,k), \beta_n(a,k))$. Thus, for example, our pair at (3.5) is the dual of the WP-Bailey pair of Andrews and Berkovich in [2]:
$$\alpha_n(a,k) = \begin{cases} \dfrac{(a, q^2\sqrt{a}, -q^2\sqrt{a}, a^2/k^2; q^2)_{n/2}}{(q^2, \sqrt{a}, -\sqrt{a}, q^2k^2/a; q^2)_{n/2}} \left(\dfrac{k}{a}\right)^n, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$
$$\beta_n(a,k) = \frac{(k,\ k\sqrt{q/a},\ -k\sqrt{q/a},\ a/k; q)_n}{(\sqrt{aq},\ -\sqrt{aq},\ qk^2/a,\ q; q)_n} \left(\frac{-k}{a}\right)^n.$$

We had initially thought that the WP-Bailey pair in Lemma 1 was the dual of the following pair of Andrews and Berkovich:

(3.7)
$$\alpha_n^{(4)}(a,k) = \frac{1-aq^{2n}}{1-a} \frac{(a, k/aq; q)_n}{(q, q^2a^2/k; q)_n} \frac{(qa^2/k; q)_{2n}}{(k;q)_{2n}} \left(\frac{k}{a}\right)^n,$$
$$\beta_n^{(4)}(a,q) = \frac{(k^2/qa^2; q)_n}{(q;q)_n}.$$

However, the dual of the pair in Lemma 1 is the pair

(3.8)
$$\alpha_n^{(5)}(a,k) = \frac{1-aq^{2n}}{1-a} \frac{(a, kq/a; q)_n}{(q, a^2/k; q)_n} \frac{(a^2/k; q)_{2n}}{(kq;q)_{2n}} \left(\frac{k}{a}\right)^n,$$
$$\beta_n^{(5)}(a,q) = \frac{1-k}{1-kq^{2n}} \frac{(qk^2/a^2; q)_n}{(q;q)_n},$$

while the dual of the pair at (3.7) is the pair

$$\alpha_n^{(6)}(a,k) = \frac{1-aq^{2n}}{1-a}\frac{(a^2/qk^2;q)_n}{(q,q)_n}\left(\frac{k}{a}\right)^n, \tag{3.9}$$
$$\beta_n^{(6)}(a,k) = \frac{(a/kq,k;q)_n}{(k^2q^2/a,q;q)_n}\frac{(qk^2/a;q)_{2n}}{(a;q)_{2n}}.$$

Inserting the new WP-Bailey pairs in any of the existing WP-Bailey chains will lead to transformations relating basic hypergeometric series. We believe the following transformations of basic hypergeometric series to be new.

Corollary 10.

(3.10)
$$ {}_{12}\phi_{11}\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, y, z, \frac{qa}{k}, \frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}, -k\sqrt{\frac{q}{a}}, \frac{kaq^{N+1}}{yz}, q^{-N} \\ \sqrt{k}, -\sqrt{k}, \frac{qk}{y}, \frac{qk}{z}, \frac{k^2}{a}, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a}, kq^{N+1}, \frac{yzq^{-N}}{a} \end{matrix}; q, q\right]$$
$$= \frac{(kq, kq/yz, aq/y, aq/z;q)_N}{(kq/y, kq/z, aq/yz, aq;q)_N}\, {}_5\phi_4\left[\begin{matrix} y, z, \frac{qa^2}{k^2}, \frac{kaq^{N+1}}{yz}, q^{-N} \\ \frac{qa}{y}, \frac{qa}{z}, aq^{N+1}, \frac{yzq^{-N}}{k}\end{matrix}; q, q\right].$$

Proof. Insert the WP-Bailey pair at (3.1) into (2.3) and set $\rho_1 = y$ and $\rho_2 = z$. □

Substituting the pair at (3.1) into (2.4) leads to the following result.

Corollary 11.

$$ {}_7\phi_6\left[\begin{matrix} k, \frac{qa}{k}, \frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}, -k\sqrt{\frac{q}{a}}, q^{-N} \\ \frac{k^2}{a}, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a}, \frac{k^2q^{-N}}{a^2}\end{matrix}; q, q\right] = \tag{3.11}$$
$$\frac{(qa/k, qa^2/k;q)_N}{(qa, qa^2/k^2;q)_N}\, {}_7\phi_6\left[\begin{matrix} \frac{qa^2}{k^2}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a^2q^{N+1}}{k}, q^{-N} \\ a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, \frac{kq^{-N}}{a}, aq^{N+1}\end{matrix}; q, q\right].$$

Corollary 12.

(3.12)
$$ {}_{14}W_{13}\left(q; y, yq, z, zq, \frac{q^2}{ad}, \frac{dq}{a}, q^2, \frac{aq^{N+2}}{yz}, \frac{aq^{N+3}}{yz}, q^{-N}, q^{1-N}; q^2, q^2\right) -$$
$$aq\frac{\left(1-\frac{d}{a}\right)\left(1-\frac{q}{ad}\right)\left(1-q^3\right)\left(1-q^{-N}\right)\left(1-y\right)\left(1-\frac{aq^{2+N}}{yz}\right)\left(1-z\right)}{\left(1-ad\right)\left(1-q\right)\left(1-\frac{aq}{d}\right)\left(1-q^{2+N}\right)\left(1-\frac{q^2}{y}\right)\left(1-\frac{q^2}{z}\right)\left(1-\frac{yzq^{-N}}{a}\right)}$$

$$\times\, {}_{14}W_{13}\left(q^3; yq, yq^2, zq, zq^3, \frac{q^3}{ad}, \frac{dq^2}{a}, q^2, \frac{aq^{N+3}}{yz}, \frac{aq^{N+4}}{yz}, q^{1-N}, q^{2-N}; q^2, q^2\right)$$

$$= \frac{\left(q^2, \frac{q^2}{yz}, \frac{aq}{y}, \frac{aq}{z}; q\right)_N}{\left(\frac{q^2}{y}, \frac{q^2}{z}, \frac{aq}{yz}, aq; q\right)_N} {}_{10}W_9\left(a; y, z, d, \frac{q}{d}, -a, \frac{aq^{N+2}}{yz}, q^{-N}; q, -a\right).$$

$$(3.13)\quad {}_7\phi_6\left[\begin{matrix} q^{5/2},\ -q^{5/2},\ y,\ \frac{q}{y},\ \frac{q^2}{ad},\ \frac{dq}{a},\ q^2 \\ q^{1/2},\ -q^{1/2},\ q^2y,\ \frac{q^3}{y},\ qad,\ \frac{q^2a}{d}\end{matrix}; q^2, a^2\right]$$

$$- \frac{a^2\left(1-\frac{d}{a}\right)\left(1-\frac{q}{ad}\right)\left(1-q^3\right)\left(1-\frac{q}{y}\right)(1-y)}{(1-ad)(1-q)\left(1-\frac{aq}{d}\right)\left(1-\frac{q^2}{y}\right)(1-qy)}$$

$$\times\, {}_7\phi_6\left[\begin{matrix} q^{7/2},\ -q^{7/2},\ yq,\ \frac{q^2}{y},\ \frac{q^3}{ad},\ \frac{dq^2}{a},\ q^2 \\ q^{3/2},\ -q^{3/2},\ q^3y,\ \frac{q^4}{y},\ q^2ad,\ \frac{q^3a}{d}\end{matrix}; q^2, a^2\right]$$

$$= \frac{(q, q^2; q)_\infty (ady, adq/y, ayq/d, q^2a/dy; q^2)_\infty}{(ab, aq/d, yq, q^2/y; q)_\infty}.$$

Proof. Insert the WP-Bailey pair at (3.3) into (2.3), set $\rho_1 = y$ and $\rho_2 = z$ and replace k with q. For (3.13), set $z = q/y$, let $N \to \infty$ and use (3.4) to sum the resulting right side. □

Corollary 13.

$$(3.14)\quad {}_5\phi_4\left[\begin{matrix} \frac{q^2}{ad}, \frac{dq}{a}, q^{-N}, q^{1-N}, q^2 \\ \frac{aq^2}{d}, adq, \frac{q^{2-N}}{a^2}, \frac{q^{3-N}}{a^2}\end{matrix}; q^2, q^2\right]$$

$$- aq\frac{(1-q^{-N})(1-q/ad)(1-d/a)}{(1-q^{2-N}/a^2)(1-ad)(1-aq/d)}\, {}_5\phi_4\left[\begin{matrix} \frac{q^3}{ad}, \frac{dq^2}{a}, q^{1-N}, q^{2-N}, q^2 \\ \frac{aq^3}{d}, adq^2, \frac{q^{3-N}}{a^2}, \frac{q^{4-N}}{a^2}\end{matrix}; q^2, q^2\right]$$

$$= \frac{(a, a^2; q)_N}{(qa, a^2/q; q)_N} {}_{10}W_9\left(a; \sqrt{q}, -\sqrt{q}, d, \frac{q}{d}, q, a^2q^N, q^{-N}; q, -a\right).$$

Proof. Insert the WP-Bailey pair at (3.3) into (2.4), and replace k with q. □

Remark: The extra q-products inserted in each of the series on the right side of (3.12) and in (3.14) are there to allow these series to be represented as ${}_{r+1}\phi_r$ or ${}_{r+1}W_r$ series.

Corollary 14.

(3.15)

$$ {}_{12}W_{11}\left(k; \frac{qa}{k}, \frac{k}{\sqrt{a}}, \frac{-k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}, -k\sqrt{\frac{q}{a}}, \sqrt{ak}q^N, -\sqrt{ak}q^N, -q^{-N}, q^{-N}; q, q^2\right) $$
$$ = \frac{\left(\frac{k}{a}, k^2q^2, -a, -aq; q^2\right)_N}{\left(\frac{a}{k}, a^2q^2, -kq, -kq^2; q^2\right)_N}\left(\frac{aq}{k}\right)^N $$
$$ \times\ {}_7\phi_6\left(\begin{matrix} iq\sqrt{a}, -iq\sqrt{a}, \frac{a^2q}{k}, \sqrt{ak}q^N, -\sqrt{ak}q^N, -q^{-N}, q^{-N} \\ i\sqrt{a}, -i\sqrt{a}, aq^{1+N}, -aq^{1+N}, \sqrt{\frac{a}{k}}q^{1-N}, -\sqrt{\frac{a}{k}}q^{1-N} \end{matrix}; q, q\right). $$

Proof. After replacing m with k and employing some simple transformations for q-products, (2.15) can be rewritten as

$$ \sum_{j=0}^{N}\frac{1-kq^{2j}}{1-k}\frac{(akq^{2N}, q^{-2N}; q^2)_j}{\left(k^2q^{2+2N}, \frac{k}{a}q^{2-2N}; q^2\right)_j}q^{2j}\beta_j(a,k) \tag{3.16} $$
$$ = \frac{\left(\frac{k}{a}, k^2q^2; q^2\right)_N}{\left(\frac{a}{k}, a^2q^2; q^2\right)_N}\frac{(-a;q)_{2n}}{(-kq;q)_{2n}}\left(\frac{aq}{k}\right)^N $$
$$ \times\sum_{j=0}^{N}\frac{1+aq^{2j}}{1+a}\frac{(akq^{2N}, q^{-2N}; q^2)_j}{\left(a^2q^{2+2N}, \frac{a}{k}q^{2-2N}; q^2\right)_j}\left(\frac{aq}{k}\right)^j\alpha_j(a,k). $$

The result follows, upon inserting the pair from Lemma 1, and rearranging. □

Inserting the new pairs in other WP-Bailey chains will lead to other, possibly new, transformations of basic hypergeometric series, but we refrain from further examples here.

4. WP-Burge Pairs

In [2], the authors termed a WP-Bailey pair $(\alpha_n(1,k), \beta_n(1,k))$ in which $\alpha_n(1,k)$ does not depend on k a *WP-Burge pair.*

One such pair that they derive in [2, (7.16)] is the pair in the following theorem. For completeness, we give an alternative proof of this result.

Theorem 2. *Define*

$$ \alpha_n(1,k) = \begin{cases} 1, & n=0, \\ q^{-n/2}+q^{n/2}, & n\geq 1, \end{cases} \tag{4.1} $$
$$ \beta_n(1,k) = \frac{(k\sqrt{q}, k; q)_n}{(\sqrt{q}, q; q)_n}q^{-n/2}. $$

Then $(\alpha_n(1,k), \beta_n(1,k))$ *satisfy* (1.1) *(with* $a=1$*).*

Proof. We begin by recalling Bailey's ${}_6\psi_6$ summation formula [13].

$$\frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,qa^2/bcde;q)_\infty}$$
$$=\sum_{n=-\infty}^{\infty}\frac{(1-aq^{2n})(b,c,d,e;q)_n}{(1-a)(aq/b,aq/c,aq/d,aq/e;q)_n}\left(\frac{qa^2}{bcde}\right)^n$$
$$=1+\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(b,c,d,e;q)_n}{(1-a)(aq/b,aq/c,aq/d,aq/e;q)_n}\left(\frac{qa^2}{bcde}\right)^n$$
$$+\sum_{n=1}^{\infty}\frac{(1-1/aq^{2n})(b/a,c/a,d/a,e/a;q)_n}{(1-1/a)(q/b,q/c,q/d,q/e;q)_n}\left(\frac{qa^2}{bcde}\right)^n,$$

where the second equality follows from the definition

$$(z;q)_{-n}=\frac{(-1)^n q^{n(n+1)/2}}{z^n(q/z;q)_n}.$$

Next, set $a=-1$, $b=-c$ and $d=-e$, so that both sums in the final expression above become equal, to get

$$\frac{(-q,q/c^2,q/e^2;q)_\infty(q^2/c^2e^2,q^2/c^2e^2,q^2;q^2)_\infty}{(q/c^2e^2;q)_\infty(q^2/c^2,q^2/c^2,q^2/e^2,q^2/e^2;q^2)_\infty}$$
$$=1+\sum_{n=1}^{\infty}\frac{(1+q^{2n})(c^2,e^2;q^2)_n}{(q^2/c^2,q^2/e^2;q^2)_n}\left(\frac{q}{c^2e^2}\right)^n,$$

or, upon replacing c^2 with c and e^2 with e,

$$\frac{(-q,q/c,q/e;q)_\infty(q^2/ce,q^2/ce,q^2;q^2)_\infty}{(q/ce;q)_\infty(q^2/c,q^2/c,q^2/e,q^2/e;q^2)_\infty} \tag{4.2}$$
$$=1+\sum_{n=1}^{\infty}\frac{(1+q^{2n})(c,e;q^2)_n}{(q^2/c,q^2/e;q^2)_n}\left(\frac{q}{ce}\right)^n.$$

Now replace q with $\sqrt{q}$, set $e=q^{-N}$, $c=kq^N$ to get, after some minor rearrangements, that

$$1+\sum_{n=1}^{N}\frac{(kq^N,q^{-N};q)_n}{(q^{1-N}/k,q^{1+N};q)_n}\left(\frac{q}{k}\right)^n(q^{-n/2}+q^{n/2})=\frac{(k\sqrt{q},q;q)_N}{(k,\sqrt{q};q)_N}q^{-N/2}$$

The result now follows from (1.1), after some simple manipulations. □

One reason this WP-Bailey pair is somewhat interesting is that the special case $k=0$ of (4.1) gives Slater's standard Bailey pair **F3** from [10]. Recall that a pair of sequences (α_n,β_n) is termed a *Bailey pair relative to x/q*, if they satisfy

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(x;q)_{n+r}}, \tag{4.3}$$

We also recall the following result of Bailey, a particular case of the "Bailey Transform", from his 1949 paper [4].

Theorem 3. *Subject to suitable convergence conditions, if the sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy* (4.3), *then*

(4.4)
$$\sum_{n=0}^{\infty}(y,z;q)_n\left(\frac{x}{yz}\right)^n\beta_n=\frac{(x/y,x/z;q)_\infty}{(x,x/yz;q)_\infty}\sum_{n=0}^{\infty}\frac{(y,z;q)_n}{(x/y,x/z;q)_n}\left(\frac{x}{yz}\right)^n\alpha_n.$$

Thus the pair at (4.1) "lifts" this standard Bailey pair to a WP-Bailey pair and lifts all the series–product identities following from **F3** to more general series–product identities containing the free parameter k.

More generally, let $(\alpha_n(a,k),\beta_n(a,k))$ be a WP-Bailey pair in which $\alpha_n(a,k)$ is independent of k. Suppose further that this pair is a "lift" of a standard Bailey pair, in the sense that setting $k=0$ in the WP-Bailey pair recovers the Bailey pair. By comparing (4.4) and (2.5) (first setting $y=\rho_1$, $z=\rho_2$ and $x=aq$ in (4.4)) we see that the two infinite series containing $\alpha_n=\alpha_n(a,k)$ are identical. This means that if particular choices of y and z in (4.4) lead to an identity of the Rogers-Ramanujan type, then the same choices for ρ_1 and ρ_2 in (2.5) will lead to a more general series–product identity containing an extra free parameter, namely k, and this more general identity will revert back to the original identity of Rogers-Ramanujan type, upon setting $k=0$.

We will investigate this phenomenon further in a subsequent paper.

References

[1] Andrews, George E. *Bailey's transform, lemma, chains and tree.* Special functions 2000: current perspective and future directions (Tempe, AZ), 1–22, NATO Sci. Ser. II Math. Phys. Chem., **30**, Kluwer Acad. Publ., Dordrecht, 2001.

[2] Andrews, George; Berkovich, Alexander *The WP-Bailey tree and its implications.* J. London Math. Soc. (2) **66** (2002), no. 3, 529–549.

[3] Bailey, W. N., *Some Identities in Combinatory Analysis.* Proc. London Math. Soc. **49** (1947) 421–435.

[4] Bailey, W. N., *Identities of the Rogers-Ramanujan type.* Proc. London Math. Soc., **50** (1949) 1–10.

[5] Bressoud, David, *Some identities for terminating q-series.* Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 2, 211–223.

[6] Gasper, George; Rahman, Mizan *Basic hypergeometric series.* With a foreword by Richard Askey. Second edition. Encyclopedia of Mathematics and its Applications, 96. Cambridge University Press, Cambridge, 2004. xxvi+428 pp.

[7] Q. Liu; X. Ma *On the Characteristic Equation of Well-Poised Baily Chains* - To appear.

[8] Mc Laughlin, James; Zimmer, Peter. *Some Applications of a Bailey-type Transformation* - submitted

[9] Singh, U. B. *A note on a transformation of Bailey.* Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 177, 111–116.

[10] Slater, L. J. *A new proof of Rogers's transformations of infinite series.* Proc. London Math. Soc. (2) 53, (1951). 460–475.

[11] Spiridonov, V. P. *An elliptic incarnation of the Bailey chain.* Int. Math. Res. Not. 2002, no. **37**, 1945–1977.

[12] M. V. Subbarao; A. Verma. *Some summations of q-series by telescoping.* Pacific J. Math. **191** (1999), no. 1, 173–182.

[13] Watson, G. N. *The Final Problem: An Account of the Mock Theta Functions.* J. London Math. Soc. **11**, 55-80, 1936.
[14] Warnaar, S. O. *Extensions of the well-poised and elliptic well-poised Bailey lemma.* Indag. Math. (N.S.) **14** (2003), no. 3-4, 571–588.

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